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BENDING AND BUCKLING ANALYSIS OF FUNCTIONALLY GRADED MICROPLATES USING ISOGEOMETRIC APPROACH

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ABSTRACT

This study aims at investigating size-dependent effects of functionally graded (FG) microplates for bending and buckling problems using isogeometric analysis (IGA). While displacement field of the microplates is established based on the refined plate theory (RPT) with four variables, the modified couple stress (MCS) theory is employed to capture small scale effects of the microplates. Bending and buckling analysis of rectangular and circular FG microplates based on these two theories are solved in the platform of isogeometric analysis. This recently developed method utilises the non-uniform rational B-splines (NURBS) functions to establish approximation functions and describe geometry domains. In addition, NURBS functions, by its nature, could satisfy high-order continuity which is required in RPT theory without any difficulty. A number of numerical examples conducting for rectangular and circular FG microplates reveal that the consideration of small scale effects is followed by the increase in plates' stiffness. Consequently, transverse deflection and buckling load of the FG microplates will be decreased and risen, respectively.

Key Words: *FG microplates; modified couple stress theory; refined plate theory; isogeometric analysis; bending and buckling analysis*

1. Introduction

Functionally graded materials (FGMs) are composite materials formed of two or more constituent phases in which material properties vary smoothly from one surface to the other. With the rapid development of technology, FGMs have been increasingly used in micro electromechanical systems (MEMS/NEMS), electrically-actuated MEMS devices, atomic force microscopes, etc. Mechanical properties of such small-scale structural devices including Young's modulus, flexural rigidity are size-dependent [6]. However, classical continuum elasticity which is scale-free theory fails to predict these size effects. Modified couple stress theory (MCST) [4, 6] appears to be one of the size-dependent theories in which only one material length scale parameter is employed. Based on the MCST, a couple of research works using various plate models have been carried out to investigate the size-dependent effects of micro FG plates, one of them is to use refined plate theory (RPT) [1]. However, RPT requires C^1-continuity of general displacements causing significant challenge to derive second derivative of deflection in the platform of finite element analysis (FEA) where C^0 elements are frequently used. Recently, a new numerical method so-called Isogeometric Analysis (IGA) which is able to deal with C^1-continuity problem without using any additional variables has been introduced by Hughes and his co-workers [2]. This method bridges the gap between Computer Aided Design (CAD) and FEA in which same basis functions generated by B-splines or non-uniform rational B-splines (NURBS) shape functions are employed to describe exact geometry and unknown variables. Since modelled geometry is exact and the number of unknown terms is not increased, it is expected that IGA would yield more accurate results with lower computational cost for RPT problems in the comparison with regular FEA [3].

In this study, the size-dependent bending and buckling behaviours of FG microplates will be investigated. While the size effects are captured using the modified couple stress theory, the four-variable refined plate theory is employed to describe displacement field. The problems are solved in the platform of the isogeometric analysis.

2. Theoretical formulation of FG microplates

Assuming that the functionally graded microplates are made of metal and ceramic varying continuously through the plates' thickness, the effective material properties of the plates are obtain following the rule of mixture or Mori-Tanaka scheme. Detail of those expressions could be found from some research in the literature [3].

According to the modified couple stress theory, the strain energy is defined as a combination of strain tensor and curvature tensor [1]. The components of the deviatoric part of the symmetric couple stress tensor m_{ij} and symmetric curvature tensor χ_{ij} are given by

$$m_{ij} = 2Gl^2\chi_{ij}; \quad \chi_{ij} = \frac{1}{2}\left(\frac{\partial\theta_i}{\partial\chi_j} + \frac{\partial\theta_j}{\partial\chi_i}\right) \tag{1}$$

where G and l are shear modulus and material length scale parameter, respectively, and the rotation vector $\boldsymbol{\theta} = \frac{1}{2}curl(\mathbf{u})$ where $\mathbf{u}$ is the displacement vector.

In order to take into account the shear deformation effects, the plate displacement field is presented with respect to the refined plate theory as

$$u\left(x,y,z\right) = u_0\left(x,y\right) - zw_{b,x}\left(x,y\right) + g\left(z\right)w_{s,x}\left(x,y\right) \tag{2a}$$

$$v\left(x,y,z\right) = v_0\left(x,y\right) - zw_{b,y}\left(x,y\right) + g\left(z\right)w_{s,y}\left(x,y\right); \quad w\left(x,y,z\right) = w_b\left(x,y\right) + w_s\left(x,y\right) \tag{2b}$$

where u_0, v_0 are membrane displacements, w_b and w_s represent bending and shear components of transverse displacement, $g(z)$ is the distribution function which is taken as $g(z) = -4z^3/(3h^2)$ [5]. The rotation vector and curvature vector are obtained by substituting the above displacement expressions into Equation 1 [1].

3. NURBS-based formulation of FG microplates

By using the NURBS basis functions, the displacement field $\boldsymbol{u}$ of the FG microplates based on the RPT and MCST could be approximated as follow [2, 3]

$$\mathbf{u}^h\left(\xi,\eta\right) = \sum_{A}^{m\times n} R_A\left(\xi,\eta\right)\mathbf{q}_A \tag{3}$$

where $m \times n$ is the number of basis functions, R_A represents two-dimensional NURBS basis functions and $\mathbf{q}_A$ denotes the vector of degrees of freedom associated with the control point A. The isogeometric finite element formulation of the bending and buckling problems are defined as

$$\mathbf{Kq} = \mathbf{F}; \quad \left(\mathbf{K} - \lambda_{cr}\mathbf{K}_g\right)\mathbf{q} = \mathbf{0} \tag{4}$$

where λ_{cr} presents buckling parameter, and $\mathbf{F}$ and $\mathbf{K}_g$ are load vector and geometric stiffness matrix, respectively [3]. The global stiffness matrix $\mathbf{K}$ of the structures is a summation of two components corresponding to the classical term $\mathbf{K}_1$ and couple stress term $\mathbf{K}_2$. The full expression of classical term $\mathbf{K}_1$ in the global stiffness matrix could be found in the work of Nguyen *et al.* [3] while the couple stress counterpart is defined as

$$\mathbf{K}_2 = \int_\Omega \left[\left\{\begin{array}{c}\mathbf{B}^{b0}_{mA}\\ \mathbf{B}^{b1}_{mA}\end{array}\right\}^T \left[\begin{array}{cc}\mathbf{X}^c & \mathbf{Y}^c\\ \mathbf{Y}^c & \mathbf{Z}^c\end{array}\right]\left\{\begin{array}{c}\mathbf{B}^{b0}_{mA}\\ \mathbf{B}^{b1}_{mA}\end{array}\right\} + \left\{\begin{array}{c}\mathbf{B}^{s0}_{mA}\\ \mathbf{B}^{s2}_{mA}\end{array}\right\}^T \left[\begin{array}{cc}\mathbf{A}^c & \mathbf{B}^c\\ \mathbf{B}^c & \mathbf{D}^c\end{array}\right]\left\{\begin{array}{c}\mathbf{B}^{s0}_{mA}\\ \mathbf{B}^{s2}_{mA}\end{array}\right\}\right] d\Omega \tag{5}$$

where

$$\mathbf{B}_{mA}^{b0} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 4R_{A,xy} & 2R_{A,xy} \\ 0 & 0 & -4R_{A,xy} & -2R_{A,xy} \\ 0 & 0 & 2\left(-R_{A,xx}+R_{A,yy}\right) & -R_{A,xx}+R_{A,yy} \end{bmatrix}, \quad \mathbf{B}_{mA}^{b1} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & -2R_{A,xy} \\ 0 & 0 & 0 & 2R_{A,xy} \\ 0 & 0 & 0 & R_{A,xx}-R_{A,yy} \end{bmatrix} \tag{6a}$$

$$\mathbf{B}_{mA}^{s0} = \frac{1}{4}\begin{bmatrix} -R_{A,xy} & R_{A,xx} & 0 & 0 \\ -R_{A,yy} & R_{A,xy} & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_{mA}^{s2} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & -R_{A,y} \\ 0 & 0 & 0 & R_{A,x} \end{bmatrix} \tag{6b}$$

and material matrices are given by

$$\left(A_{ij}^c, B_{ij}^c, D_{ij}^c\right) = \int_{-h/2}^{h/2} \left(1, g'(z), \left[g'(z)\right]^2\right) G_{ij} dz; \left(X_{ij}^c, Y_{ij}^c, Z_{ij}^c\right) = \int_{-h/2}^{h/2} \left(1, g''(z), \left[g''(z)\right]^2\right) G_{ij} dz \tag{7}$$

4. Numerical examples

In this section, the bending and buckling analysis of the FG microplates will be conducted based on the RPT, modified couple stress, and the proposed IGA approach. While the square plates are used to investigate the bending behaviours, buckling analysis is conducted by considering circular ones. The Al/Al_2O_3 is chosen to be the metal-ceramic FG material whose properties are of $E_m = 70$ GPa, $E_c = 380$ GPa, $\rho_m = 2707$ kg/m^3, $\rho_c = 3800$ kg/m^3, and $\nu_m = \nu_c = 0.3$. For bending analysis, the fully simply-supported (SSSS) microplates subjected to sinusoidally

Table 1: Non-dimensional central deflection $\bar{w}$ and critical buckling load $\bar{P_{cr}}$ of Al/Al_2O_3 microplates

l/h	$\bar{w}$, square plates			$\bar{P_{cr}}$, circular plates		
	a/h	Present	Ref. [5]	h/R	Simple BC	Clamp BC
0	5	0.6688	0.6688	0.1	11.8496	38.3789
	100	0.5625	0.5625	0.2	11.4869	34.8238
0.2	5	0.5505	0.5468	0.1	12.2075	44.2232
	100	0.4689	0.4689	0.2	11.8392	40.3906
0.6	5	0.2288	0.2224	0.1	13.4377	90.9639
	100	0.2012	0.2011	0.2	12.9976	84.8716
1	5	0.1060	0.1017	0.1	14.1319	184.4163
	100	0.0939	0.0939	0.2	13.7383	173.7056

distributed load are analysed considering the aspect ratio a/h, material length scale ratio l/h, and material index $n = 1$. In order to verify the availability of the proposed theories and approach, the results are compared to analytical solutions reported by Thai and Kim [5]. As can be seen in the Table 1 showing the non-dimensional central deflection $\bar{w} = 10wE_ch^3/(q_0L^4)$, the present results, in general, are in good agreement with those of the previous authors [5]. For thick plates ($a/h = 5$), the solutions from both approaches which are very close to each other for $l/h = 0$ but they are discrepancy as l/h and n become larger. However, these discrepancies vanish as the thin plates become thinner, i.e. $a/h = 100$. It could also be observed that, for all cases, the increase in material length scale ratio l/h which rises the plate stiffness leads to the decrease of central deflection.

With regard to buckling analysis, non-dimensional biaxial critical buckling which is defined as $\bar{P_{cr}} = 12(1-\nu_m^2)P_{cr}R^2/(E_mh^3)$ of the Al/Al_2O_3 circular microplates for simple support and clamped boundary conditions are tabulated in Table 1. Similar to the case of bending analysis, due to the increase in the plates' stiffness, the critical buckling load grows up as the material parameter length scale l/h rises. Corresponding to $h/R = 0.2, l/h = 0.4$, and $n = 1$, the first two buckling mode shapes along with their buckling load values of the clamped microplates are depicted in the Figure 1. The mode shapes are all scaled up for the illustration purposes.

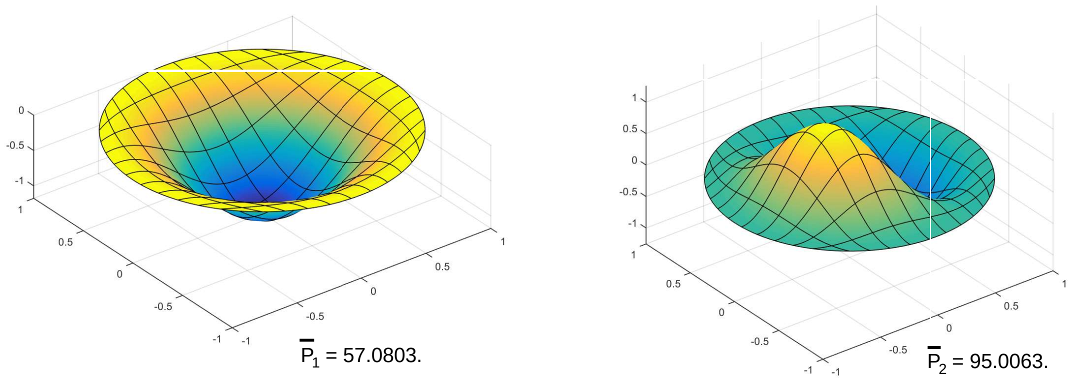

Figure 1: First two buckling mode shapes of clamped plates.

5. Concluding remarks

The size-dependent bending and buckling behaviours of the functionally graded microplates have been investigated. The size effects are considered by the modified couple stress theory. The four-variable refine plate theory is employed to describe the displacement field of the plates. The formulation of the square and circular microplates using the chosen theories which require C^1-continuity and exact geometry are then successfully integrated and solved in the platform of isogeometric analysis. The numerical results which are in good agreement with those exist in the literature reveal that the size-dependent investigation using modified couple stress theory result in an increase in the plates' stiffness. Consequently, the deflection and buckling load of the microplates decrease and increase, respectively.

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