

Prism solid-shell with heterogonous and hierarchical approximation basis

*Lukasz Kaczmarczyk¹, Zahur Ullah¹ and Chris Pearce¹

¹School of Engineering, University of Glasgow

*Lukasz.Kaczmarczyk@glasgow.ac.uk

ABSTRACT

A solid-shell element which does not possess rotational degrees of freedom (DOFs) and which is applicable to thin plate/shell problems is considered. The element approximation is constructed in prisms, where displacements on the upper and lower surfaces are approximated in the global coordinate system. In addition, two other fields are defined in the shell natural (local) coordinate system that represent the components of the displacement vector in both the current shell normal direction and the current shell tangent plane. To each field, an arbitrary order of approximation can be defined, and all fields reproduce a complete and conforming polynomial approximation basis for the solid prism element. It is not necessary to augment the formulation with an assumed natural strain (ANS) field or enhanced assumed strain (EAS) field or to use reduced integration, making the element ideally suited for geometrically and physically nonlinear problems.

Key Words: solid shell, large deformations, hierarchical approximation

1. Introduction

In standard thin shell formulations, the approximation through the thickness is assumed to be linear or higher order, i.e. the normals to the mid-surface in the initial configuration remain straight but not normal during the deformation. In the proposed formulation the thickness of the element is not constant and the normal stress in through-thickness direction is considered. For this reason, the kinematics of this solid-shell element is richer than the kinematic assumed in Kirchhoff - Love plate theory. In formulating this element, we do not aim to reproduce classical shell kinematic, thereby avoiding the problem of element locking. This work builds on a substantial body of published work by a number of different authors, most notably [4]. However this implementation proposes a new formulation exploiting hierarchical, heterogeneous and anisotropic approximation spaces.

2. Hierarchical approximation in prism

In constructing the approximation space in a prism we apply a similar procedure to that shown in [2] for tetrahedra. Nodal basis functions using barycentric coordinates are given by

$$\phi^v = \lambda_v, \quad {}_L\phi^v = \phi^v \zeta, \quad {}_U\phi^v = \phi^v (1 - \zeta) \quad (1)$$

where right subscript ${}_L\phi^v$ and ${}_U\phi^v$ indicates shape functions on lower and upper triangles respectively and λ_v denotes the barycentric coordinates. Convective coordinate $\zeta \in [0, 1]$ is the coordinate through the prism thickness. $\zeta = 0$ defines the lower prism triangle and $\zeta = 1$ the upper prism triangle.

The edge hierarchical approximation basis is constructed as follows

$$\beta_{0i} = \lambda_0 \lambda_i, \quad \phi_{i,t}^{e_t} = \beta_{0i} L_l(\lambda_i - \lambda_j), \quad {}_L\phi_{i,t}^{e_t} = \phi_{i,t}^{e_t} \zeta, \quad {}_U\phi_{i,t}^{e_t} = \phi_{i,t}^{e_t} (1 - \zeta), \quad (2)$$

where i and j are nodal indices on a triangle. L_l is the Legendre polynomial of order l . If p is the order of the polynomial for the triangle, then $0 \leq l \leq p - 2$ and the number of DOFs on an edge is $p - 1$. The triangle approximation basis is constructed by

$$\beta_{0ij} = \lambda_0 \lambda_i \lambda_j, \quad \phi_{i,m}^t = \beta_{0ij} L_l(\lambda_0 - \lambda_i) L_m(\lambda_0 - \lambda_j) \zeta, \quad {}_L\phi_{i,m}^t = \phi_{i,m}^t \zeta, \quad {}_U\phi_{i,m}^t = \phi_{i,m}^t (1 - \zeta) \quad (3)$$

If p is the order of the polynomial on a triangle, then $0 \geq l, m, l + m \geq p - 2$ and number of DOFs on triangle is $(p - 1)(p - 2)/2$.

The edge through-thickness basis function is given by

$$\beta_{00} = \lambda_0 \lambda_0, \quad \phi_l^{e_q} = \beta_{00} \zeta (1 - \zeta) L_l (2\zeta - 1) \quad (4)$$

where λ_0 is the barycentric coordinate for the node of the triangle to which the edge through thickness is adjacent. If p is the order of the polynomial in the prism, then $0 \geq l \geq p - 2$ and the number of DOFs on edge is $p - 1$. The quadrilateral through thickness basis function is

$$\beta_{0i} = \lambda_0 \lambda_i, \quad \phi_{l,m}^q = \beta_{0i} \zeta (1 - \zeta) L_l (\lambda_0 - \lambda_i) L_m (2\zeta - 1) \quad (5)$$

where 0 and i indicate nodes on opposite corner nodes of a quadrilateral with its own canonical numbering. If p is the order of the polynomial in the prism, then $0 \geq l, m, l + m \geq p - 4$ and the number of DOFs on the quadrilateral is $(p - 3)(p - 2)/2$. The bubble prism basis functions are given by

$$\beta_{0i} = \lambda_0 \lambda_i \lambda_j, \quad \phi_{l,m,k}^P = \beta_{0ij} \zeta (1 - \zeta) L_l (\lambda_0 - \lambda_i) L_m (\lambda_0 - \lambda_j) L_k (2\zeta - 1) \quad (6)$$

where 0, i, j are indices of nodes on the triangle. If p is polynomials order in prism, then $0 \geq l, m, k, l + m + k \geq p - 5$ and number of DOFs of the prism is $(P - 5)(P - 4)(P - 3)/6$.

3. Geometry approximation in reference configuration

A pragmatic approach is to generate the input mesh surface using a standard mesh generator. In case of non-planar surfaces, the geometry can be defined by 6-noded or higher-order triangles. The geometry defined by the surface mesh is then projected using a hierarchical basis onto the prism. Thus all approximations for the shell geometry using a hierarchal basis shown below is directly inferred from the input triangular mesh. The only additional information required is the shell thickness.

The reference position of points on the shell mid-surface, in the global Cartesian coordinate system, is given on the triangular mesh by

$${}^M \mathbf{Z}(\xi, \eta) = {}^3 \phi^v(\xi, \eta) \underline{\mathbf{Z}}^v + {}^3 \phi^{e_t}(\xi, \eta) \underline{\mathbf{Z}}^{e_t} + {}^3 \phi^t(\xi, \eta) \underline{\mathbf{Z}}^t = \sum_{g=v, e_t, t} {}^3 \phi^g(\xi, \eta) \underline{\mathbf{Z}}^g \quad (7)$$

where $\underline{\mathbf{Z}}^{v, e_t, t}$ are DOFs on vertices, edges and triangles. Left upper-script ${}^3 \phi$ indicate that it is matrix of base approximation functions for vector field, where size of vector is three. In addition we define on the surface mesh a field of unit length director vectors [6], as follows

$${}^M \mathbf{V}(\xi, \eta) = \sum_{g=v, e_t, t} {}^3 \phi^g(\xi, \eta) \underline{\mathbf{V}}^g \quad (8)$$

Finally reference position vector in prism element is given by in global Cartesian coordinate system

$$\mathbf{Z}(\xi, \eta, \zeta) = {}^M \mathbf{Z}(\xi, \eta) + a {}^M \mathbf{V}(\xi, \eta) (\zeta - \frac{1}{2}) = \sum_{s=L, U} \sum_{g=v, e_t, t} {}^3 \phi_s^g(\xi, \eta, \zeta) \underline{\mathbf{Z}}_s^g \quad (9)$$

where a is the shell thickness, which may not necessarily be constant.

4. Curvilinear systems in reference and current configuration

The vector coefficients in the Cartesian coordinate system are Z^I and z^i for the current and reference configuration, respectively. For simplicity, we use both coordinate systems with the same base vectors and origin. In addition we use local curvilinear coordinate base for reference configuration, where vectors of the base are \mathbf{E}_α and vector of covariant coefficients in that base are X^A . The field of \mathbf{E}_α is consequently approximated using hierarchical approximation as follows

$${}^M \mathbf{E}_{0,1}(\xi, \eta) = \left\{ \sum_{g=v, e_t, t} {}^3 \phi^g(\xi, \eta) \underline{\mathbf{E}}_{0,1}^g \right\} \mathbf{I} \quad {}^M \mathbf{E}_2(\xi, \eta) = \left\{ \text{Spin}[{}^M \mathbf{E}_0(\xi, \eta)] {}^M \mathbf{E}_1(\xi, \eta) \right\} \mathbf{I} \quad (10)$$

where $\text{Spin}[\cdot]$ is a spin operator acting as a vector product and $\underline{\mathbf{E}}_{0,1}^s$ are coefficients of the approximation functions.

The local curvilinear coordinate base in the current configuration, convected by the motion of the shell mid-surface is given by

$$\mathbf{e}_a = e_i^a \mathbf{i}_i = {}^M F_I^i E^{AI} \delta_A^a \mathbf{i}_i \quad (11)$$

where ${}^M \mathbf{F}_I^i(\xi, \eta, \zeta)$ is component of the gradient of deformation, i.e. for the discretized system, the gradient of deformation is additively decomposed into a part resulting from through-thickness DOFs and a part resulting from DOFs on the triangles. The latter part is used to push the base vectors.

5. Displacements and physical equations

In the following approach, for convenience, we use the local shell coordinate system to evaluate the physical equations. The Cartesian global coordinate system is used to express coefficients of DOFs on entities adjacent to the upper and lower triangles. For DOFs on entities through the shell thickness, i.e. edges through the thickness, quads and the prism itself, the current curvilinear (convected) shell coordinate system is used to express coefficients of the displacement vector.

The position of a material point in the current configuration is given by

$$z^a \mathbf{i}_a = Z^J \mathbf{I}_J + u^J \mathbf{I}_J + v e_2^J \mathbf{I}_J \quad (12)$$

where displacements are additively decomposed into a global and through thickness component. The displacement \mathbf{u} is approximated in the global coordinate system as

$$u^J(\xi) = \sum_{s=L,U} \sum_{g=v,e,t} {}^3 \phi_s^g(\xi) \underline{\mathbf{u}}_s^{g,J} = \left[{}^3 \bar{\phi}(\xi) \right] \underline{\mathbf{u}}^J. \quad (13)$$

Note that displacement DOFs in vector $\underline{\mathbf{u}}$ are in global Cartesian coordinate system, thus this solid shell element could be assembled with other tetrahedral elements without the need for any linking/transfer elements. The displacement $\mathbf{v} = v e_2^J \mathbf{I}_J$ is approximated in the local conservative system following global mid-surface deformation. The vector of displacements through the thickness is given by

$$v(\xi) = \sum_{g=e,q,p} {}^1 \phi^g(\xi) \underline{\mathbf{v}}^{g,a} = \left[{}^1 \bar{\phi}(\xi) \right] \underline{\mathbf{v}}. \quad (14)$$

Note that the vector of values of DOFs $\underline{\mathbf{v}}$ is in the local, curvilinear current (conservative) coordinate system. Those degrees of freedom are associated with edges, quads and volume of prism itself, thus are inside volume of shell structure and are not adjacent to any other DOFs except those of the considered shell.

Consequently the vector of internal forces is

$$\mathbf{f}_u^{\text{int}}(\underline{\mathbf{u}}, \underline{\mathbf{v}}) = \int_{\Omega} \left[\nabla_{Z^I} {}^3 \bar{\phi} \right] P_i^I(\underline{\mathbf{u}}, \underline{\mathbf{v}}) d\Omega \quad \text{and} \quad \mathbf{f}_v^{\text{int}}(\underline{\mathbf{u}}, \underline{\mathbf{v}}) = \int_{\Omega} \left[\nabla_{X^A} {}^1 \bar{\phi} \right] P_2^A(\underline{\mathbf{u}}, \underline{\mathbf{v}}) d\Omega, \quad (15)$$

where P_i^I and P_a^A are Piola - Kirchhoff tensors in the Cartesian and local convected curvilinear system. Consequently $\mathbf{f}_u^{\text{int}}$ and $\mathbf{f}_v^{\text{int}}$ are internal force vectors associated with DOFs adjacent to triangles (upper and lower) in a global cartesian system and DOFs adjacent to edges through the thickness, quads and prism itself in a local curvilinear coordinate system.

In the formulation presented here, the current curvilinear system follows the shell deformation given by DOFs on upper and lower triangle. This, in some sense, leads to a co-rotational like formulation [3]. As result it is noted that the tangent stiffness is non-symmetric. This observation is consistent with that made by Crisfield [3], who adapted the co-rotational formulation whereby the coordinate systems is rotated when shell is deformed. However, numerical experiments have shown that for the problems addressed here, like for the co-rotational formulation, the tangent stiffness matrix becomes symmetric as the iterative procedure reaches equilibrium, thus the matrix can be symmetrized without deteriorating the rate of convergence.

6. Example

In the following examples we present two classical tests of a pinched cylinder. The reference solutions provided by [5] are reproduced. However, exploring the flexibility of the method, increasing the approximation order in the shell plane or shell thickness, we are able to find a softer, converged solution, see Figure 1.

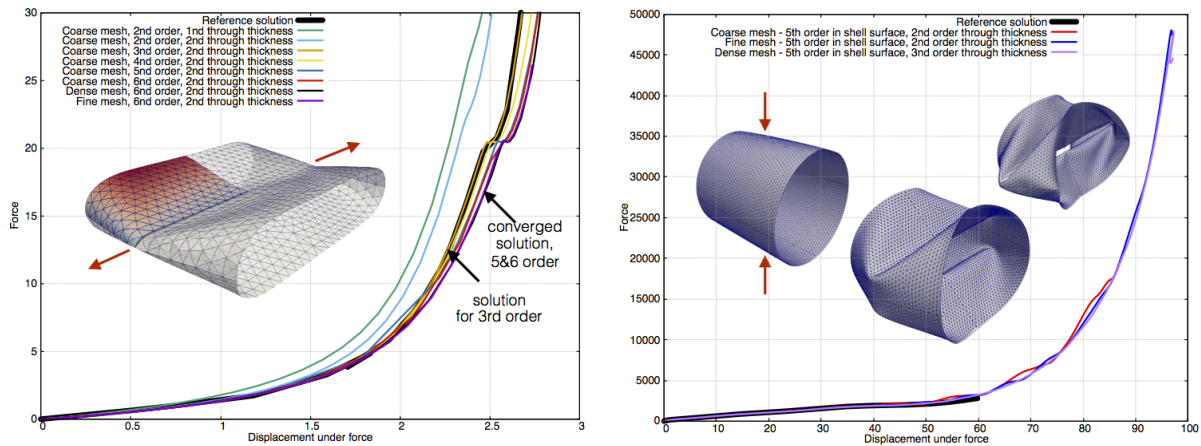


Figure 1: Pinched cylinder without diaphragm tension on the left and with diaphragm compression on the right. See [5] for more information about geometry, material parameters and reference solution. Analysed meshes are available from [1].

7. Conclusions

The solid-shell element presented here has a number of properties. First, element DOFs do not possess rotations, such that element could be used in conjunction with classical solid elements without the need for any additional transfer elements. Second, the approximation basis is hierarchical. Such an approximation allows for efficient construction of iterative solvers tailored for hp-adaptive code. Third, the approximation basis is heterogeneous, that is an arbitrary approximation order can be set independently for each geometrical entity, i.e. edge, triangle, quad or prism. Fourth, local approximation of membrane displacements and normal displacements through the thickness are independent from each other. Finally, the physical equation for 3d solid can be used in the local shell coordinate system.

References

- [1] MoFEM finite element code. <http://userweb.eng.gla.ac.uk/lukasz.kaczmarczyk/MoFem/html/index.html>.
- [2] Mark Ainsworth and Joe Coyle. Hierarchic finite element bases on unstructured tetrahedral meshes. *International Journal for Numerical Methods in Engineering*, 58(14):2103–2130, 2003.
- [3] Michael A Crisfield. A consistent co-rotational formulation for non-linear, three-dimensional, beam-elements. *Computer methods in applied mechanics and engineering*, 81(2):131–150, 1990.
- [4] R Hauptmann and K Schweizerhof. A systematic development of solid shell element formulations for linear and non linear analyses employing only displacement degrees of freedom. *International Journal for Numerical Methods in Engineering*, 42(1):49–69, 1998.
- [5] Saman Hosseini, Joris JC Remmers, Clemens V Verhoosel, and René Borst. An isogeometric solid-like shell element for nonlinear analysis. *International Journal for Numerical Methods in Engineering*, 95(3):238–256, 2013.
- [6] Theodore Sussman and Klaus Jurgen Bathe. 3d-shell elements for structures in large strains. *Computers and Structures*, 122:2 – 12, 2013. Computational Fluid and Solid Mechanics 2013 Proceedings Seventh {MIT} Conference on Computational Fluid and Solid Mechanics.