Proceedings of the 24th UK Conference of the
Association for Computational Mechanics in Engineering
31 March - 01 April 2016, Cardiff University, Cardiff

A reduced order model for criticality problems in reactor physics varying control rod settings

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ABSTRACT

In this paper we apply model order reduction to the criticality problem in reactor physics. Starting from the eigenvalue problem associated with the multigroup neutron diffusion equation we build a reduced order model from solutions for different control rod positions using proper orthogonal decomposition. The reduced system of equations can be solved at a fraction of the cost of the full system, yielding a solution for any given control rod height. We demonstrate this procedure for a section of a fuel assembly.

Key Words: *Reduced order modelling; proper orthogonal decomposition; multigroup neutron diffusion; eigenvalue problem; reactor criticality*

1. Introduction

Using reduced order models (ROMs) [1] when solving parameterised problems is known to be beneficial, as the problem size can be reduced by several orders of magnitude whilst retaining reasonable solution accuracy. In this paper, ROM is applied to the eigenvalue problem arising from the study of criticality in nuclear reactors. The method proposed is based on proper orthogonal decomposition (POD) [2] and the method of snapshots [3]. There has been a considerable amount of attention in POD from within the fluid dynamics community, having been successfully applied to a range of applications such as the mixing of fluid layers [4] and ocean models [5]. However, its application to solving reactor physics problems is less common. Notable exceptions are [6] which recasts the criticality eigenvalue problem as a time dependent problem, and [7] which develops a control-oriented model of nuclear reactor spatial kinetics.

The growth of the neutron population in a reactor is one of the most important characteristics of a nuclear system, knowledge of which determines the state of the reactor encapsulated in the effective multiplication factor k^{eff}. Ideally, for a particular reactor, k^{eff} needs to be calculated for any position of the control rods and any temperature distribution. As these change constantly, solving a demanding computation for each control rod height and temperature distribution is not practical. Reduced order modelling is ideally suited to such a problem.

The following section describes the multigroup diffusion equation and the formulation of the criticality problem. In section 3 the reduced order model is described. Section 4 presents some preliminary results, and the final section concludes and identifies future work.

2. Governing Equations

Under operating conditions in a reactor, the multigroup neutron diffusion equation can be used to model the neutron population. The multigroup discretisation splits up the energy into a number of so-called energy groups determined by a discrete set of values $E_G < E_{G-1} < \cdots < E_1 < E_0$. Each group has an associated scalar flux solution, ϕ_g, and a set of material parameters. The behaviour of the neutrons is described by the macroscopic cross-sections corresponding to fission, scattering and absorption, denoted Σ_g^f, $\Sigma_{g'\to g}^s$, Σ_g^a respectively, where the subscripts g and g' indicate a particular group. The cross-sections represent the probabilities that fission, scattering or absorption will occur. The parameter D_g

which controls how the neutrons diffuse through the domain, depends on the cross-sections according to $D_g(x) = \left(3(\Sigma_g^a(x) + \Sigma_g^s(x))\right)^{-1}$. Criticality is a measure of the growth or decay of the neutron population in a system over successive neutron generations in a fission chain reaction. In order to study criticality, the governing equation is cast in the form of an eigenvalue problem by introducing the effective multiplication factor, k^{eff}. If neutrons removed from the system exactly balance those created, the system is adjudged critical ($k^{\text{eff}} = 1$). If more neutrons are produced than removed then the system is super-critical ($k^{\text{eff}} > 1$). If fewer neutrons are produced than removed the system is sub-critical ($k^{\text{eff}} < 1$).

For group g, the criticality eigenvalue problem of the multigroup neutron diffusion equation is

$$-\nabla \cdot (D_g(x)\nabla\phi_g(x)) + \Sigma_g^a(x)\,\phi_g(x) - \sum_{\substack{g'=1 \\ g'\neq g}}^{G} \Sigma_{g'\to g}^s(x)\,\phi_{g'}(x) = \frac{\chi_g}{k^{\text{eff}}} \sum_{g'=1}^{G} \nu_{g'}\Sigma_{g'}^f(x)\,\phi_{g'}(x), \qquad (1)$$

where G is the total number of groups, χ_g is the probability that fission will result in a neutron being born in group g, ν_g is the average number of neutrons produced per fission event and k^{eff} is the effective multiplication factor. There are two types of boundary conditions considered here: reflective boundary conditions where $D_g\nabla\phi_g(x)\cdot n = 0$, and void boundary conditions $-\frac{1}{2}D_g\nabla\phi_g(x)\cdot n = \frac{1}{4}\phi_g(x)$.

Considering the case of two energy groups, we can make assumptions that (i) any neutrons created will be born into the higher energy group (ie. $\chi_1 = 1$ and $\chi_2 = 0$), (ii) upscatter from the low energy group to the fast energy group will not occur (ie. $\Sigma_{2\to1}^s = 0$) . For the full model, a Galerkin finite element discretisation is applied in space, so the solution is expanded in terms of piecewise polynomial functions (shape functions), $\phi_g(x) = N_j(x)\hat{\phi}_{g,j}$, the governing equation is weighted by the shape functions and integrated over the domain Ω. The divergence theorem is applied in order to impose boundary conditions, resulting in

$$A\,\phi \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \frac{1}{k^{\text{eff}}} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} \equiv \frac{1}{k^{\text{eff}}}\,B\,\phi \qquad (2)$$

where

$$\left(A_{gg}\right)_{ij} = \int_V \left(-D_g\nabla N_i\cdot\nabla N_j + \Sigma_g^a N_i N_j\right) dV + \frac{1}{2}\int_{S^{\text{void}}} N_i N_j\, dS, \quad \text{for} \quad g = 1,2 \qquad (3)$$

$$(A_{21})_{ij} = \int_V -\Sigma_{1\to2}^s N_i N_j\, dV \qquad (4)$$

$$\left(B_{gg'}\right)_{ij} = \chi_g \int_V \nu_{g'}\Sigma_{g'}^f N_i N_j\, dV \quad \text{for} \quad g,g' = 1,2\,. \qquad (5)$$

The dependence of the shape functions on x is no longer indicated explicitly.

3. ROM

Reduced order modelling aims to solve a computationally costly problem in a cheaper way, by using information from a series of pre-calculated solutions. For a range of values of a parameter, high resolution solutions are sought, known as snapshots. A singular value decomposition (SVD) is then applied to the matrix of snapshots which results in a set of orthogonal basis functions and a set of singular values. The magnitudes of the singular values indicate the importance of each basis function therefore not all need be retained in the POD basis if their contribution is deemed sufficiently small relative to a predefined tolerance. Once the POD basis has been found the discretised governing equation is projected onto the reduced order space where it is then solved for desired values of the parameter. The parameter varied in this paper is the control rod height, which takes a set of values from h_1 (fully inserted) to h_n (fully withdrawn), yielding n snapshots. The fraction of energy of the original information which is contained by m POD basis functions is determined by the singular values σ_k:

$$\text{energy of the first } m \text{ modes} = \frac{\sum_{k=1}^{m}\sigma_k^2}{\sum_{k=1}^{n}\sigma_k^2}\,. \qquad (6)$$

For each control rod height, the matrices A and B are projected by the m POD basis functions from finite element (FE) space to the reduced space by the following

$$A_k^{\mathrm{POD}} = R^{\mathrm{T}} A_{h_k} R \tag{7}$$

where the POD basis functions make up the columns of R and A_{h_k} is given by equation (2) evaluated for a control rod height of h_k. In order to solve a problem in the reduced order space, for a control rod height of $\tilde{h} \in [h_i, h_{i+1}]$, a linear interpolation is performed between A_i^{POD} and A_{i+1}^{POD} such that

$$A^{\mathrm{POD}} = \omega A_i^{\mathrm{POD}} + (1-\omega) A_{i+1}^{\mathrm{POD}} \qquad \text{where} \quad \omega = \frac{h_{i+1} - \tilde{h}}{h_{i+1} - h_i} \,. \tag{8}$$

B^{POD} is constructed similarly. Now we are able to assemble and solve the reduced order system

$$A^{\mathrm{POD}} \phi^{\mathrm{POD}} = \lambda^{\mathrm{POD}} B^{\mathrm{POD}} \phi^{\mathrm{POD}} \,. \tag{9}$$

Finally we project the flux solution from the reduced space into the FE space by calculating $R\,\phi^{\mathrm{POD}}$.

4. Results

Part of a fuel assembly is modelled to test the ROM procedure described in the previous section. The domain is $[-1.845, 1.845] \times [-1.845, 1.845] \times [-5, 5]$. There are 8 fuel pins and one control rod of radii 0.48, surrounded by a moderator which in this case is water (see figure 1).

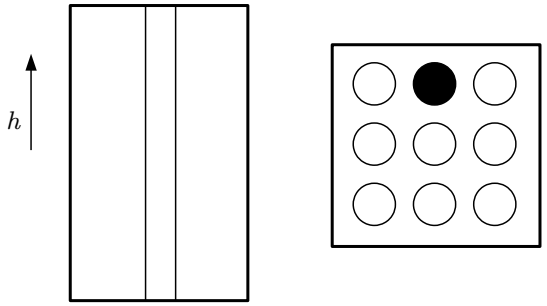

Figure 1: *Left,* part of a fuel assembly showing the region where the control rod is located. *Right,* view from above of the assembly showing fuel pins and control rod (shaded). (Not to scale.)

The cross-sections were generated using the ANSWERS software, WIMS, and some values are reported in table 1. The mesh used has 431883 tetrahedral elements and 76712 nodes. The snapshots were taken for a control rod at heights of $\{-5, -3, -1, 1, 3, 5\}$. Each energy group had 6 POD basis functions so the reduced order system is of size 12×12. For the same heights the reduced order model gives values of k^{eff} which agree with those of the snapshots to within 6 decimal places (see figure 2). For so-called unseen values of the control rod height, the reduced order model predicts k^{eff} to a reasonable accuracy. The computational time to solve the full order model for one parameter was 379.8 seconds, whereas the reduced order model solves in less than 0.1 seconds.

		Σ_1^a	Σ_2^a	$\Sigma_{1\to1}^s$	$\Sigma_{1\to2}^s$	$\Sigma_{2\to2}^s$	Σ_1^f	Σ_2^f
fuel	$300K$	6.003E-03	9.583E-02	3.364E-01	1.336E-03	4.048E-01	3.333E-03	4.994E-02
	$2000K$	6.042E-03	9.499E-02	3.370E-01	1.340E-03	4.079E-01	3.333E-03	4.945E-02
moderator	$300K$	1.447E-04	6.597E-03	3.395E-01	5.957E-02	1.281E+00	0.0	0.0
	$2000K$	1.447E-04	6.584E-03	3.395E-01	5.954E-02	1.279E+00	0.0	0.0

Table 1: Cross-sections for the fuel and moderator at a given temperature are interpolated from the values above.

5. Final remarks

This paper has applied model order reduction to the criticality eigenvalue problem associated with the multigroup neutron diffusion equation. Snapshots were taken for different values of control rod height. When comparing the eigenvalue of the snapshots generated by the full model with that of the reduced order model, the agreement is excellent. For unseen problems, the agreement is reasonable. The full

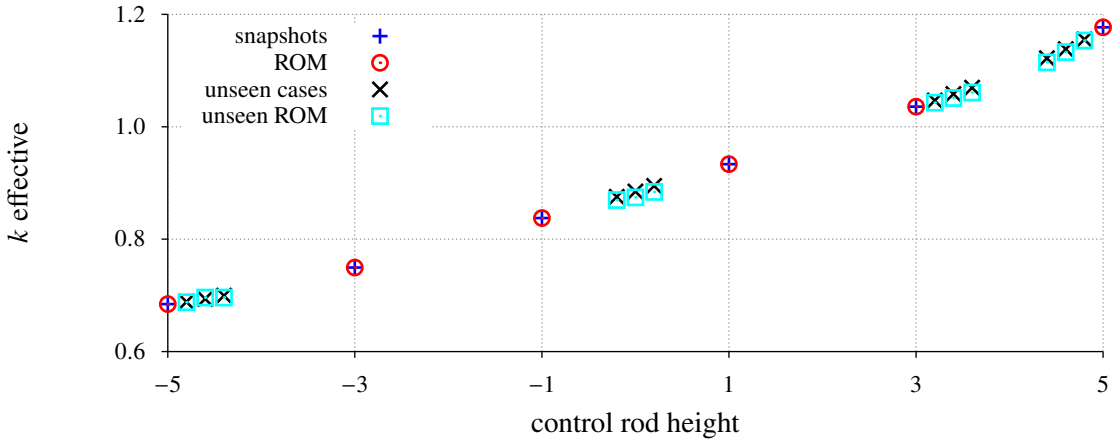

Figure 2: Two comparisons, first of k^{eff} from the snapshots (full model) with k^{eff} from the reduced order model as the control rod height varies (blue crosses and red circles), and second of k^{eff} for full model solutions with k^{eff} of unseen values of the control rod height predicted by ROM (black crosses and cyan squares).

order model has 153424 unknowns whereas the reduced order problem has just 12 and solves in a fraction of the time.

Future work will involve (i) quantifying the error between the reduced order model solution and the full model for unseen problems and (ii) introducing temperature to the ROM as a parameter, as the cross-sections depend on this.

Acknowledgements

The authors gratefully acknowledge financial support from the Defence Academy at HMS Sultan. Dr Andrew Buchan acknowledges support from the EPSRC Fellowship EP/M022684/1.

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