

Solution of three dimensional transient heat diffusion problems using an enriched finite element method

*M. Iqbal¹, M.S. Mohamed¹, O. Laghrouche¹, H. Gimperlein^{2,3}, M. Seaid⁴, J. Trevelyan⁴

¹Institute for Infrastructure and Environment, Heriot Watt University, Edinburgh, EH14 4AS

²Maxwell Institute for Mathematical Sciences and Department of Mathematics, Heriot Watt University, Edinburgh, EH14 4AS

³Department of Mathematics, University of Paderborn, Warburger Str. 100, 33098 Paderborn, Germany

⁴School of Engineering and Computing Sciences, Durham University, Durham, DH1 3LE

*mi130@hw.ac.uk

ABSTRACT

We propose an enriched finite element method for the solution of three dimensional transient heat diffusion problems. The steep gradient of the solution is captured by a suitable enrichment of the finite element space. We use multiple exponential functions to enrich the solution space. This results in an efficient method, as is illustrated in an example problem with known analytical solution. In this problem we compare the proposed scheme with a standard FEM discretization. It is shown that the enriched FEM strongly reduces the necessary number of degrees of freedom to achieve a prescribed accuracy.

Key Words: *partition of unity method; 3D heat diffusion problems; enrichment functions*

1. Introduction

For decades FEM has been used for solving complex problems in many engineering and scientific applications. This method has shown significant accuracy and robustness for complex problems governed by steady-state PDEs. Solving time-dependent PDEs on complex geometries is still a substantial task for FEM especially for 3D problems. To overcome these difficulties various generalized and meshless numerical methods have been introduced in the recent decades. Partition of unity method PUFEM is one of the important subclass of these methods which was introduced by Melenk and Babuska [1].

The Partition of Unity Method is used for a wide variety of engineering and scientific applications. Shadi *et al* [2] solved two dimension transient heat diffusion problem using FEM and PUFEM. The authors compared results for both the methods and showed the efficiency of PUFEM compared to FEM. A full description of the use of PUFEM for different heat transfer applications can be found in their work. In the current work we exploit the PUFEM to solve 3D transient diffusion problem and show its efficiency as compared to FEM. In the proceeding section we present the formulation of the transient diffusion problem with selected initial and boundary conditions followed by its weak formulation. In Section 3, we present partition of unity method to solve the problem considered with some numerical results presented in Section 4. In Section 5 some concluding remarks are presented.

2. Weak formulation and numerical approximation

Given an open bounded domain $\Omega \subset \mathbb{R}^2$ with boundary Γ and a given time interval $]0, T]$, we consider the transient heat diffusion equation

$$\frac{\partial u}{\partial t} - \lambda \Delta u = f(t, \mathbf{x}), \quad \text{in }]0, T] \times \Omega \quad (1)$$

where $\mathbf{x} = (x, y, z)^T$ are the spatial coordinates, t is the time variable, $\lambda > 0$ is the diffusion coefficient and $f(t, \mathbf{x})$ represents the effects of internal heat sources/sinks. We consider an initial condition

$$u(t = 0, \mathbf{x}) = U_0(\mathbf{x}), \quad \mathbf{x} \in \Omega \quad (2)$$

where $U_0(\mathbf{x})$ is a prescribed initial field. We impose Robin-type boundary condition

$$\frac{\partial u}{\partial n} + hu = g, \quad \text{in }]0, T] \times \Gamma \quad (3)$$

Here \mathbf{n} denotes the outward unit normal on the boundary Γ , and $h \geq 0$ is the heat convection coefficient on Γ , and g represents boundary sources.

To solve equation (1)-(3) numerically, the time interval is divided into N_t subintervals $[t_n, t_{n+1}]$ with duration $\Delta t = t_{n+1} - t_n$ for $n = 0, 1, \dots, N_t$ and then discretized it using an implicit scheme

$$\frac{u^{n+1} - u^n}{\delta t} - \lambda \nabla^2 u^{n+1} = f(t_{n+1}, \mathbf{x}) \quad (4)$$

This can be rearranged as

$$-\nabla^2 u^{n+1} + ku^{n+1} = F \quad (5)$$

where F and k are defined as

$$F = k(\delta t f(t_{n+1}, \mathbf{x}) + u^n), \quad k = \frac{1}{\lambda \delta t}$$

To solve equation (5) with FEM we first multiply the equation with a weighting function, W , and then integrate over Ω

$$-\int_{\Omega} W \nabla^2 u^{n+1} d\Omega + \int_{\Omega} W k u^{n+1} d\Omega = \int_{\Omega} W F d\Omega \quad (6)$$

Applying the divergence theorem, substituting the initial and boundary conditions and taking $g=0$, the final weak form is given by

$$\int_{\Omega} (\nabla W \cdot \nabla u^{n+1} + W k^2 u^{n+1}) d\Omega + \int_{\Gamma} W h u^{n+1} d\Gamma = \int_{\Omega} W F d\Omega + \int_{\Gamma} W g d\Gamma \quad (7)$$

3. Solving the problem with FEM and PUFEM

To solve the weak form (7) with FEM, the domain Ω is approximated as a set of elements where the temperature over each element is approximated using the nodal values and polynomial shape functions

$$u = \sum_{i=1}^n N_i u_i \quad (8)$$

In the PUFEM these nodal values u_i are written as combination of enrichment functions. Here we propose using the following global exponential basis functions to enrich the solution space

$$G_q(\mathbf{x}) = \frac{e^{-\left(\frac{R_0}{C}\right)^q} - e^{-\left(\frac{R_c}{C}\right)^q}}{1 - e^{-\left(\frac{R_c}{C}\right)^q}}, \quad q = 1, 2, \dots, Q \quad (9)$$

With $R_0 := |\mathbf{x} - \mathbf{x}_c|$ being the distance from the function control point x_c to the point x . The constants $R_c = \sqrt{\frac{14}{1.195}}$ and $C = \frac{1}{1.195}$ controls the shape of the exponential function G_q .

Considering the aforementioned enrichment, the nodal values might be rewritten as

$$u_i = \sum_{q=1}^Q A_j^q \frac{e^{-\left(\frac{R_0}{C}\right)^q} - e^{-\left(\frac{R_c}{C}\right)^q}}{1 - e^{-\left(\frac{R_c}{C}\right)^q}} \quad (10)$$

Using (10) to rewrite (8) we get

$$u = \sum_{j=1}^M \sum_{q=1}^Q A_j^q N_j \frac{e^{-\left(\frac{R_0}{C}\right)^q} - e^{-\left(\frac{R_c}{C}\right)^q}}{1 - e^{-\left(\frac{R_c}{C}\right)^q}} \quad (11)$$

Where FEM is now used to find the unknowns A_j^q instead of the nodal values, u_i .

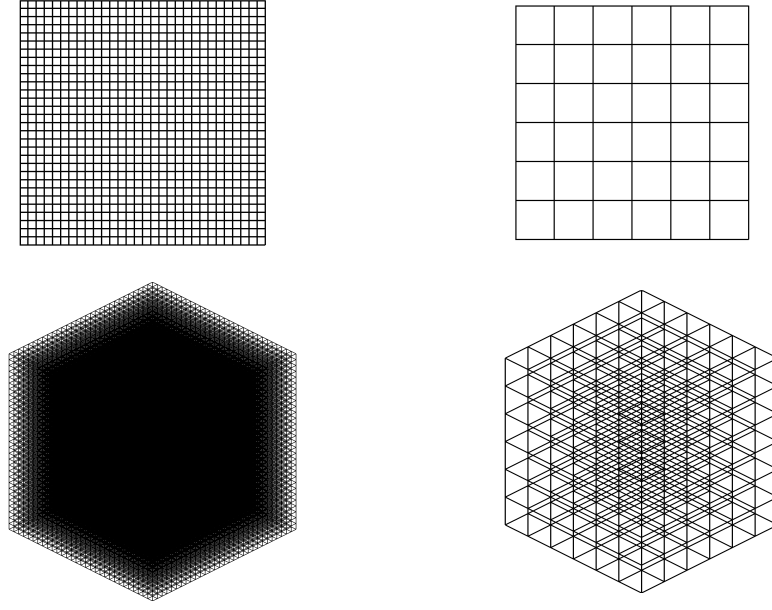


Figure 1: 3D mesh used for FEM (left) and PUFEM (right)

4. Numerical Results

In this section, we present results for the transient heat diffusion problem given by (1) - (3) using the classical FEM approach and the PUFEM approach. As a test example with consider a diffusion problem in 3D domain Ω defined by $(\mathbf{x} \in \Omega ; 0 \leq \mathbf{x} \leq 2)$. For the proposed problem, reaction term $f(t, \mathbf{x})$, the boundary function g and the initial condition $u_0(\mathbf{x})$ are chosen such that the exact solution is given by

$$U(\mathbf{x}, t) = x^{20}(2-x)^{20}y^{20}(2-y)^{20}z^{20}(2-z)^{20}(1 - e^{-\lambda t}) \quad (12)$$

where t is the time variable and $\mathbf{x} = (x, y, z)^T$ are the spatial coordinates. To quantify the error in both FEM and PUFEM, we use the l_2 norm error defined as

$$l_2 = \frac{\|u - U\|_{L^2(\Omega)}}{\|U\|_{L^2(\Omega)}} \times 100 \quad (13)$$

where u and U are the numerical and exact solutions respectively. Our aim is to compare the results of the proposed problem using the standard FEM and the PUFEM. For both the FEM and PUFEM computations we use the parameter $h=1$, the heat diffusion coefficient $\lambda=0.1$ and the time step value $\Delta t = 0.001$. All integrals over Ω are evaluated numerically using Gaussian quadrature with 20 integration points in each direction for PUFEM and 2 integration points for FEM. A direct solver is used to solve the resulting system of equations. For convergence of solution, we considered h -refinement for FEM and Q refinement for PUFEM. In case of FEM, we started with a coarse mesh of 1000 elements having total 1331 degrees of freedoms (DoFs) with which we get l_2 error of 8.7 % at $t=0.05$. We refined the mesh gradually until we get l_2 error of 0.62 % with 27000 elements having total 29791 DoFs. For the PUFEM we used a fixed mesh of 216 elements and four different number of enrichment functions $Q = 3, 4, 5$ and 6 having total 1029, 1372, 1715 and 2058 DoFs respectively. In case of PUFEM we get a comparable l_2 error of 0.69 % with only 1715 DoFs. Figure 1 shows the meshes used to get a comparable error for both FEM and PUFEM.

Figure 2 compares the convergence of FEM and PUFEM at different time intervals. On graph the log of DoFs is plotted on the abscissa while on ordinate l_2 error is plotted on logarithmic scale. It is clear from the graphs that PUFEM converges much faster than FEM and gives better l_2 error with lesser DoFs.

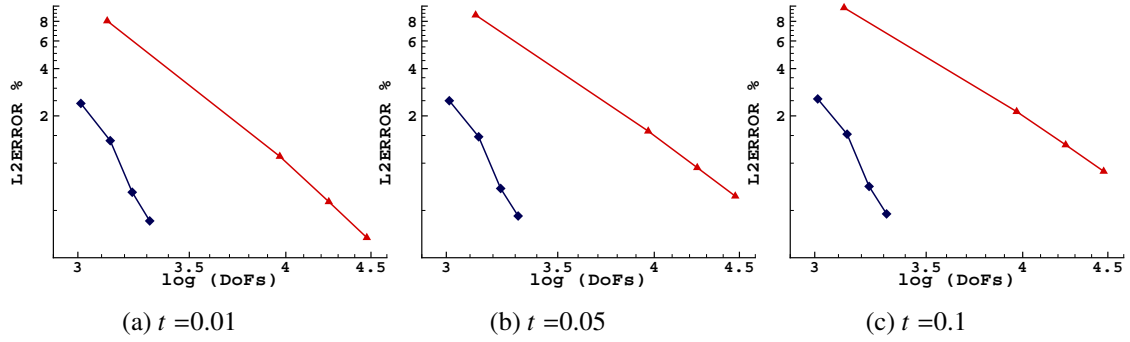


Figure 2: L2 error for FEM and PUFEM with increasing no of DoFs

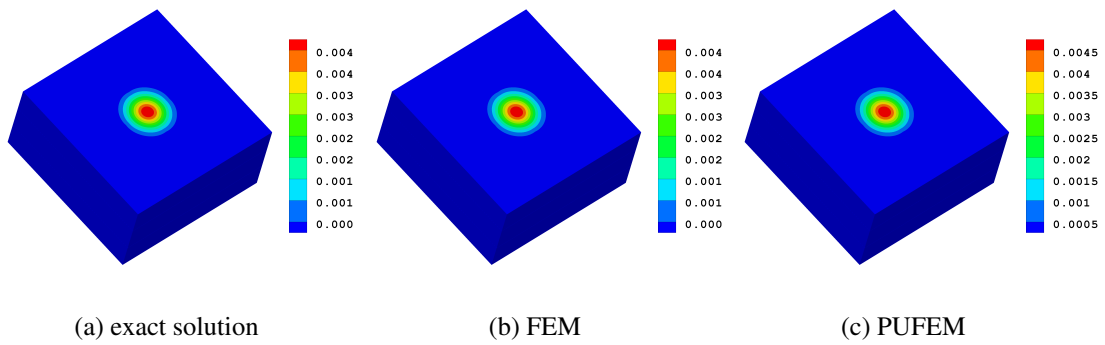


Figure 3: Temperature distribution in the middle of domain at $t = 0.05$

Figure 3 shows the temperature distribution at $t=0.05$ for the exact, FEM and PUFEM solution with 4 enrichment functions. For both FEM and PUFEM we are getting the same temperature trends as that of the exact solution but in case of PUFEM the total DoFs used to get the solution is only a small fractions of that used to obtained the solution with FEM.

5. Conclusions

We used FEM and PUFEM to solve time-dependent heat diffusion problem in 3D domain. The solution space was enriched with an approximate solution describing the heat diffusion decay. For a specified no of DoFs, we calculated l_2 error for both the methods. We concluded that for a comparable accuracy PUFEM requires less DoFs than FEM. Numerical results show that with PUFEM the total DoFs are less than 10 % of that used with FEM. Due to its higher accuracy and requiring less computational resources, the PUFEM can be used as an attractive alternative for the solution of heat diffusion problems.

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