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A PARTITION-OF-UNITY BOUNDARY ELEMENT METHOD FOR TRANSIENT WAVE PROPAGATION

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ABSTRACT

We propose a time domain partition-of-unity boundary element method for wave propagation problems at high frequency. Travelling waves are included as enrichment functions into a space-time boundary element solver. We present some first numerical experiments with this method for high frequencies, and discuss the algorithmic challenges, with a view towards engineering applications.

Key Words: *boundary element method; time domain integral equations; partition of unity method; computational acoustics; traffic noise*

1. Introduction

Boundary element methods provide an efficient, extensively studied numerical scheme for time-independent or time-harmonic scattering and emission problems. Unlike finite element discretisations, they reduce the computation from the three dimensional domain to its two dimensional boundary. Recently, boundary elements have been explored for the simulation of transient phenomena, with applications e.g. to environmental noise [2] or electromagnetic scattering [5]. For the wave equation, time-dependent boundary element methods (TDBEM) were first analysed by Bamberger and Ha-Duong [1].

On the other hand, for time-harmonic wave propagation partition-of-unity, finite and boundary element methods (PUFEM / PUBEM) have emerged as a practically efficient method to achieve engineering accuracy in spite of the numerical pollution at high frequencies [4]. More recently, first results towards time-dependent finite elements with partition-of-unity enrichment in space have been obtained in [3].

This work presents a time domain partition-of-unity boundary element method based on enrichments in space and time. It is the first investigation of a space-time enriched method, here applied to a time-dependent integral equation. The method extends the above works for time-harmonic wave propagation to truly transient problems and, as a space-time Galerkin method, can be proven to be numerically stable and convergent. Practically, it includes travelling plane-wave enrichment functions into an h-version time domain boundary element procedure.

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2. Problem description and time domain PUBEM

This work considers transient sound radiation problems in the exterior of a scatterer Ω^-, where Ω^- is a bounded polygon with connected complement $\Omega = \mathbb{R}^3 \setminus \Omega^-$. The acoustic sound pressure field $u(t, \mathbf{x})$ due to an incident field or sources on $\Gamma = \partial\Omega$ satisfies the linear wave equation for $t \in \mathbb{R}$:

$$c^{-2}\partial_t^2 u(t,\mathbf{x}) - \Delta u(t,\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in \Omega, \quad u(t,\mathbf{x}) = f(t,\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma, \quad u(t,\mathbf{x}) = 0 \quad \text{for } t \leq 0. \tag{1}$$

Here c is the wave velocity, and in the following we set $c = 1$ for simplicity. A single-layer ansatz for u,

$$u(t,\mathbf{x}) = \int_\Gamma \frac{\phi(t-|\mathbf{x}-\mathbf{y}|,\mathbf{y})}{2\pi|\mathbf{x}-\mathbf{y}|} ds_y, \tag{2}$$

results in an equivalent weak formulation of (1) as a coercive integral equation of the first kind: Find ϕ such that for all ψ

$$\int_0^\infty \int_\Gamma (V\phi(t,\mathbf{x}))\partial_t\psi(t,\mathbf{x})\ ds_x\ d_\sigma t = \int_0^\infty \int_\Gamma f(t,\mathbf{x})\partial_t\psi(t,\mathbf{x})\ ds_x\ d_\sigma t\ , V\phi(t,\mathbf{x}) = \int_\Gamma \frac{\phi(t-|\mathbf{x}-\mathbf{y}|,\mathbf{y})}{2\pi|\mathbf{x}-\mathbf{y}|} ds_y, \tag{3}$$

with $d_\sigma t = e^{-2\sigma t} dt$. A theoretical analysis requires $\sigma > 0$, but practical computations use $\sigma = 0$ [1, 2].

We propose a time-dependent boundary element method to solve (3), based on numerical approximations by travelling plane waves:

$$\phi_{h,\Delta t} = \sum_i c_i \phi_i, \quad \text{where } \phi_i(t,\mathbf{x}) = \widetilde{\Lambda}_i(t)\Lambda_i(\mathbf{x})\cos(\omega_i(t-t_i) - \mathbf{k}_i\cdot\mathbf{x} + \sigma_i)\ . \tag{4}$$

Here $\omega_i = |\mathbf{k}_i|$, $\sigma_i \in \{0, \frac{\pi}{2}\}$, Λ_i a piecewise polynomial shape function in space and $\widetilde{\Lambda}_i$ a corresponding shape function in time. A first work by Ham and Bathe uses space-enriched FEM for waves in 2d [3].

We obtain a numerical scheme for the weak formulation (3): Find $\phi_{h,\Delta t}$ such that for all $\psi_{h,\Delta t}$

$$\int_0^\infty \int_\Gamma (V\phi_{h,\Delta t}(t,\mathbf{x}))\partial_t\psi_{h,\Delta t}(t,\mathbf{x})\ ds_x\ dt = \int_0^\infty \int_\Gamma f(t,\mathbf{x})\partial_t\psi_{h,\Delta t}(t,\mathbf{x})\ ds_x\ dt\ . \tag{5}$$

From $\phi_{h,\Delta t}$, the sound pressure $u_{h,\Delta t}$ is obtained in Ω by evaluating the integral in (2) numerically. Equation (5) leads to a linear system of equations in space-time, Figure 1, where the stiffness matrix

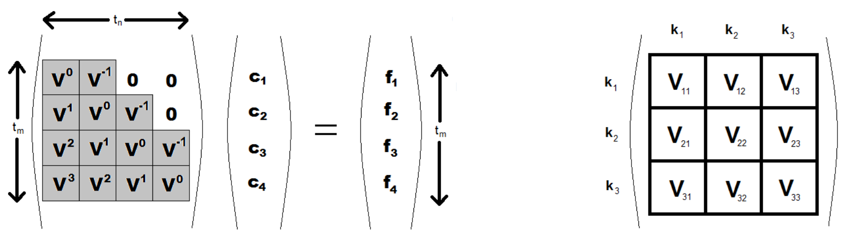


Figure 1: Full PUBEM space-time system and decomposition of the matrices V^j.

$$\mathbf{V}_{mn} = \int_0^\infty \int_\Gamma \int_\Gamma \frac{\phi_m(t-|\mathbf{x}-\mathbf{y}|,\mathbf{y})}{2\pi|\mathbf{x}-\mathbf{y}|}\partial_t\psi_n(t,\mathbf{x})\ ds_y\ ds_x\ dt$$

has a block-banded structure corresponding to the time steps. Each of the time-step blocks decomposes into blocks for the individual enrichments $\mathbf{k}_i$.

A main challenge is the accurate assembly of $\mathbf{V}_{mn}$. After an analytical evaluation of the time integral, the y integral requires integration over geometrically complicated intersections of triangles with light cone shells, with a singular integrand $|\mathbf{x}-\mathbf{y}|^{-1}$. It is evaluated in polar coordinates with a geometrically-graded hp-composite Gauss quadrature [2]. A regular Gauss quadrature is used for the x integral.

Note that the method (5), as a Galerkin method, minimises the energy $E[\phi_{h,\Delta t}] = E(\mathbf{c}) = \frac{1}{2}\mathbf{c}\cdot\mathbf{V}\mathbf{c} - \mathbf{F}\cdot\mathbf{c}$. Stability and convergence are therefore guaranteed, at least for $\sigma > 0$ or sufficiently small times [1, 2].

3. Numerical experiments

Example 1: For a regular icosahedron Γ (Figure 2) of diameter 2 and centered in (0, 0, 0), we use the right hand side $f(t,\mathbf{x}) = \exp(-25/t^2)\cos(\omega_f t - \mathbf{k}_f\mathbf{x})$, a plane wave with $\mathbf{k}_f$=(1.5, 3, 8.5) which is smoothly turned on for times [0,5]. The partition-of-unity TDBEM approximation is compared to h-TDBEM results with 1280 triangles and constant CFL ratio = 0.19. Figure 2 depicts the reference solution ϕ at times 3.8, 4.2 and 4.6 and Figure 3a the PU solution in the centroid of a triangle. The PU TDBEM uses a mesh of 20 triangles and n enrichment functions in each triangle, for $n \leq 15$ and $\Delta t = 0.1, 0.2$.

We quantify the numerical error by studying the convergence of the energy $E[\phi_{h,\Delta t}]$.

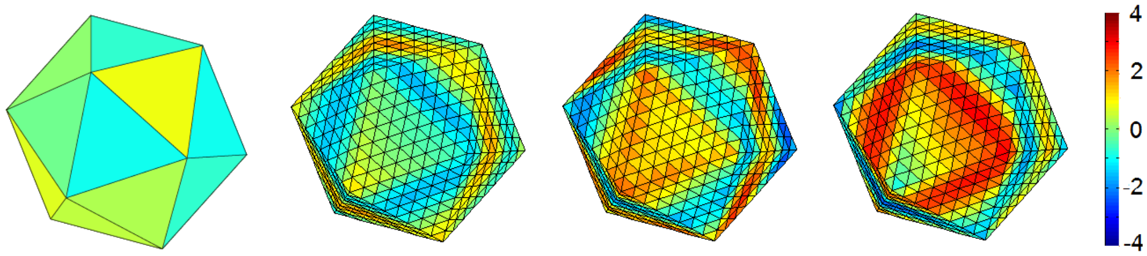
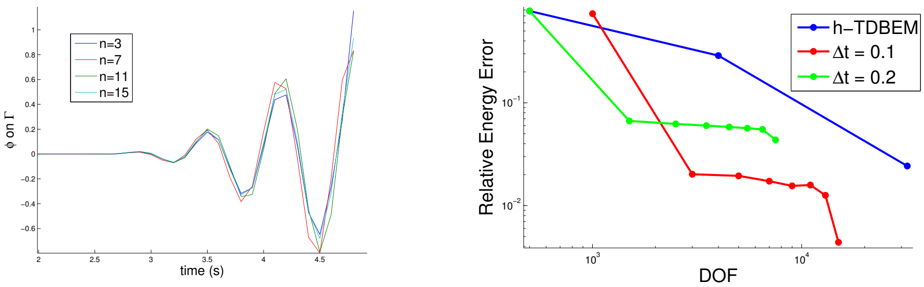


Figure 2: Example 1 - meshes for PU (20 triangles) and h-method (1280 triangles), density at $t = 3.8, 4.2, 4.6$.



(a) (b)

Figure 3: Example 1 - a) density ϕ, b) relative error in energy: h-method, PU with $\Delta t = 0.1, 0.2$.

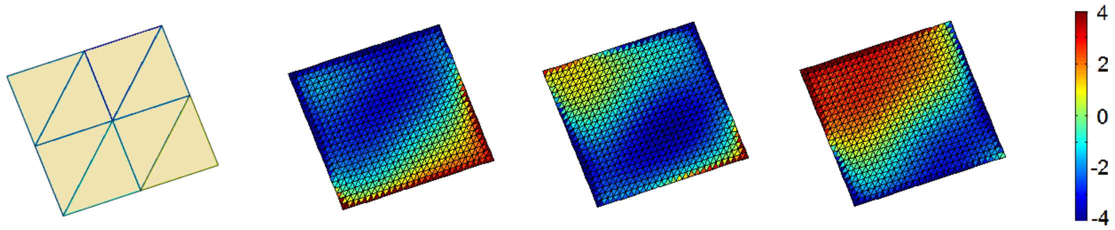
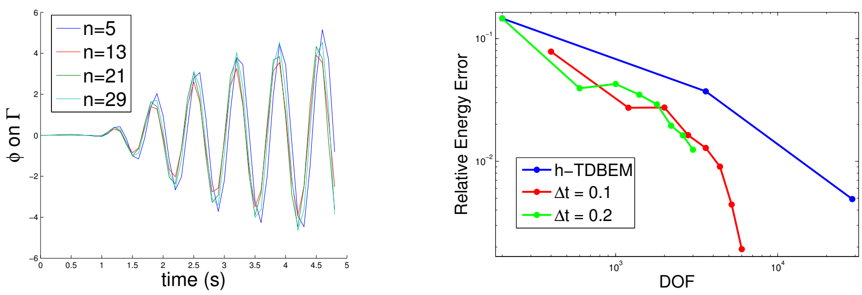


Figure 4: Example 2 - meshes for PU (8 triangles) and h-method (1250 triangles), density at $t = 3.8, 4.2, 4.6$.



(a) (b)

Figure 5: Example 2 - a) density ϕ, b) relative error in energy: h-method, PU with $\Delta t = 0.1, 0.2$.

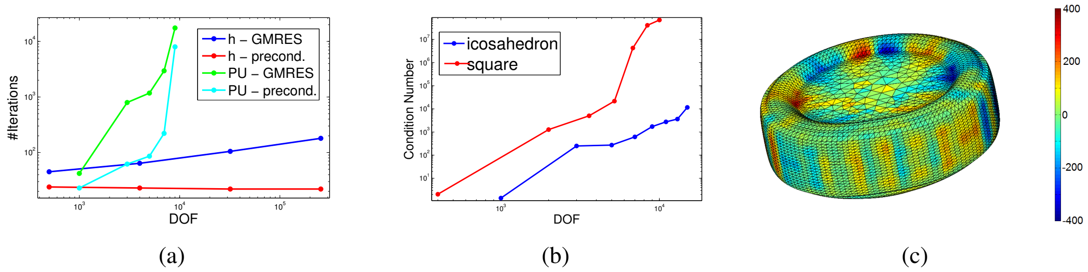


Figure 6: a) GMRES vs. preconditioned GMRES, b) condition number of $\mathbf{V}^0$ for PU, c) density ϕ on car tyre.

Figure 3b shows a comparison of convergence in the energy, between our h-TDBEM and PU-TDBEM. The PU method reduces the degrees of freedom by a factor up to 8.

Example 2: We now consider a screen $\Gamma = [0, 0.5]^2 \times \{z = 0\}$ (Figure 4). We compare a PU method on 8 triangles with $n \leq 30$ enrichment functions to an h-TDBEM with 1250 triangles and CFL ratio 0.56. We use $f(t, \mathbf{x}) = \exp(-4/t^2)\cos(\omega_f t - \mathbf{k}_f \mathbf{x})$, with $\mathbf{k}_f$=(1.5, 3, 8.5) for times [0,5].

Figure 5a shows the density ϕ at the centroid of a triangles, Figure 4 the h-TDBEM density at 3.8,4.2,4.6. Figure 5b depicts the comparison of convergence in energy for h-TDBEM and PU-TDBEM, for Δt = 0.1,0.2. For larger systems, we observe a significant reduction of the degrees of freedom for PU.

Since the memory requirements are too large for solving the large space-time linear system in Figure 1, we use an iterative GMRES solver which allows us to refer implicitly to the memory-intensive stiffness matrix via matrix-vector products only. We have developed a preconditioner for h-TDBEM, which requires less than 25 iterations independent of degrees of freedom (Figure 6a). For PU, the number of iterations is reduced. As known for time-independent PU methods, the stiffness matrix exhibits high condition numbers, here up to 10^8 on the square (Figure 6b).

4. Conclusions and Outlook

This work presents a first step towards space-time enriched methods for the wave equation, here for boundary elements. Even for finite elements such space-time enriched methods are just beginning to be explored. They present a promising approach to efficiently achieve engineering accuracy for rapidly oscillating solutions, with applications from imaging to sound radiation.

Our preliminary results for PU TDBEM already save a factor of 8 for the degrees of freedom compared to the h-method for accuracies as low as a fraction of a percent. We expect further improvements with a more thorough analysis of the time-enrichment, as well as the possibility of using larger time steps.

Outlook: Future work will investigate optimal time enrichments and preconditioning of PU TDBEM. One motivation comes from the sound radiation of tyres [2], Figure 6c.

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