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Implicit essential boundaries in the Material Point Method

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ABSTRACT

The Material Point Method (MPM) is a numerical boundary value problem (BVP) solver developed from particle-in-cell (PIC) methods that discretises the continuum into a set of material points. Information at these material points is mapped to a background Eulerian grid which is used to solve the governing equations. Once solved information is updated at the material points and these points are convected through the grid. The background grid is then reset, allowing the method to easily handle problems involving large deformations without mesh distortion. However, imposition of essential boundary conditions in the (MPM) is challenging when the physical domain does not conform to the background grid. In this research, an implicit boundary method (IBM), based on the work of Kumar et al. [1], is proposed to ensure that essential boundary conditions are satisfied in elastostatic MPM problems.

Key Words: *material point method; particle method; implicit boundary method; boundary conditions*

1. Introduction

A new modelling framework is under development at Durham University for problems associated with the ploughing of the seabed, an activity required for installation of offshore energy infrastructure (e.g. wind, tidal and marine) [1]. EPSRC have funded this research project (grant refs EP/M000397/1 and EP/M000362/1) with experimental work at Dundee University to validate the computational models developed at Durham [2]. The computational framework is based around the Material Point Method, first proposed in the 1990s [3], chosen because it is capable of modelling problems in which very large deformations occur, without the need for expensive remeshing and mapping of history variables.

The MPM for solid mechanics is described in detail in a number of references, usually in an explicit format (e.g. see [4]) although it is also possible to work with a fully implicit formulation (e.g. [5, 6]). Rather than present a lengthy derivation of the method here, the MPM is explained by reference to Figure 1. A problem domain is discretised as a set of distributed material points. In each load step, information carried at the points is interpolated to a surrounding grid of finite elements, larger than the problem domain itself. The BVP is solved on this grid using standard finite elements and the solution mapped back to the material points, which are translated into their new positions. The grid is then discarded and a new one provided for the next load step. In this way (a mixture of Lagrangian and Eulerian approaches) the issue of mesh entanglement is entirely avoided, while the individual pieces of the formulation use standard and robust finite element technology.

2. The Implicit Boundary Method

A key issue in the use of the MPM, and one not covered adequately in the existing literature, is the imposition of essential boundary conditions. Only if the background grid edge is coincident with the location of an essential boundary condition is a trivial solution to this problem possible and in the majority of modelling situations this is not possible to ensure. Burla and Kumar [7] addressed a similar problem with standard finite elements and developed a method for imposing essential boundary conditions away from finite element edges, based on ideas proposed over 50 years ago [8], using a trial function for displacement $\boldsymbol{u}$ defined as

$$\boldsymbol{u} = \boldsymbol{D}\boldsymbol{u}^g + \boldsymbol{u}^a, \tag{1}$$

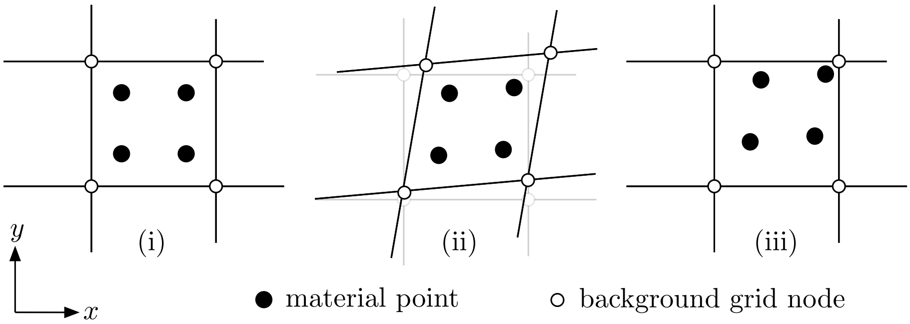

Figure 1: Steps in the Material Point Method

where $\boldsymbol{D}$ are Dirichlet functions, the components of which vanish on essential boundaries leading to the imposition of essential boundary conditions specified by $\boldsymbol{u}^a$, which could, of course, be homogeneous. The Dirichlet functions operate on the grid variable $\boldsymbol{u}^g$ which is defined in standard ways using finite element or other approaches. The same Dirichlet functions are used to construct test functions for substitution into a weak form of the PDE to be solved. The implicit boundary method for finite elements described in Burla and Kumar [7] is restricted to boundaries parallel to one or other of the coordinate axes. As part of the current project we have extended the implicit boundary method both to the MPM and also for the case of inclined boundaries and a few details and validation are presented here with a detailed description to appear in a forthcoming paper.

Figure 2 shows a straight inclined boundary in a 2D frame. For this boundary, a suitable expression for

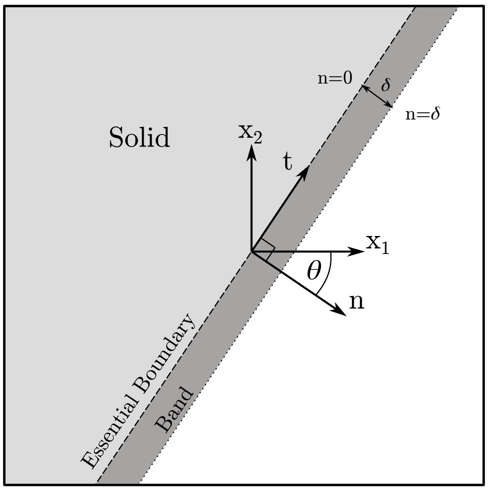

Figure 2: An implicit, inclined essential boundary showing the band width δ.

the Dirichlet function is

$$D = \frac{2n}{\delta} - \frac{n^2}{\delta^2}, \tag{2}$$

where δ defines the width of a narrow band adjacent to the boundary over which the Dirichlet function varies from zero to unity. In the implicit boundary MPM, the weak form leads to stiffness matrices associated with grid elements, which can be decomposed as follows

$$\boldsymbol{K}_e = \boldsymbol{K}_1 + (\boldsymbol{K}_2 + \boldsymbol{K}_2^T) + \boldsymbol{K}_3. \tag{3}$$

$\boldsymbol{K}_1$ is the standard local stiffness matrix of the part of the element within the solid region, while $\boldsymbol{K}_2$ and $\boldsymbol{K}_3$ are present only if an essential boundary crosses the element. These matrices are obtained from integration of matrix triple products similar to standard element stiffness matrices but with $\boldsymbol{B}$

matrices containing products of shape functions, Dirichlet functions and derivatives of both, and with transformations to fit inclined boundaries. Other approaches to dealing with this problem, e.g. non matching meshes and embedded boundary problems [9, 10], often requires the use of more complex techniques such as Lagrange multipliers, penalty methods or variations which do not have the advantage of being close to simple finite element formulations as here.

3. Validation

To demonstrate the implementation of this implicit boundary method in the MPM, results are presented for a very simple model problem. A 2D plane strain square domain of linear elastic material (100 units square) is subjected to uniaxial compression orthogonal to one side with “roller" boundary conditions on the other three sides. This problem was tested for various inclinations of the square domain while maintaining a structured background grid (Figures 3, 6 and 9). The problem domain was initially discretized by a set of material points each of which centred on a 10 unit square domain. Any void region was then truncated from material point domains lying on the border of the solid continua, and the associated material point was repositioned to the new centroid of the domain. The initial volume associated with every material point was then determined from the area of its domain. The background grid on which the elastic problem was solved consisted of a regular grid of bilinear quadrilateral elements (each 10 units square). The relative displacement and stress errors were computed and are plotted in Figures 4 & 5 (

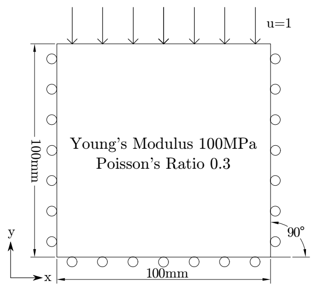

Figure 3: 90° Model

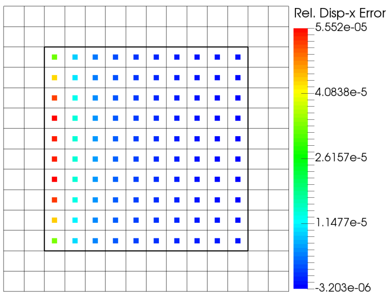

Figure 4: Relative Error in Displacement - 90° Model

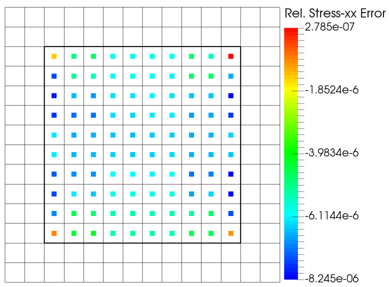

Figure 5: Relative Error in σ_{yy} - 90° Model

90° model), 7 & 8 (45° model), 10 & 11 (30° model) respectively. 5.55e-5 and 8.25e-6 relative errors in displacements and stress were obtained in the 90° model when compared to the analytical solution. The relative error in displacements and stresses reduced to 1.06e-3 and 9.9e-4 in the 45° model, and 3.14e-3 and 4.48e-3 in the 30° model respectively.

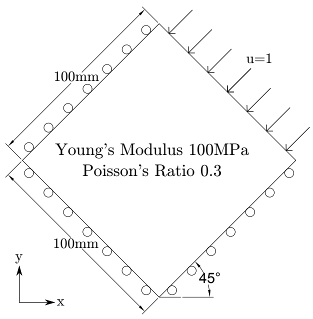

Figure 6: 45° Model

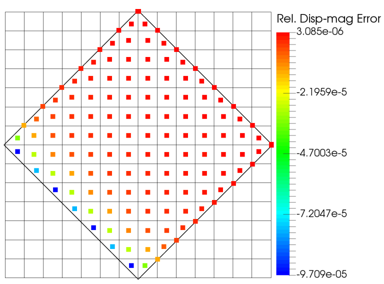

Figure 7: Relative Error in Displacement - 45° Model

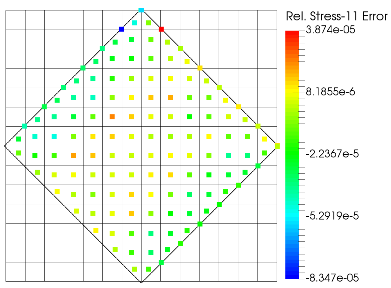

Figure 8: Relative Error in σ_1 - 45° Model

4. Conclusions

A method for imposing essential boundary conditions in the MPM without adjusting the grid or layout of material points is demonstrated in this paper. It takes a simple idea previously applied to coordinate-parallel boundaries crossing finite elements, and further develops it to model inclined essential boundaries (e.g. rollers) within an exciting new numerical methods. Ongoing work will take this into 3D.

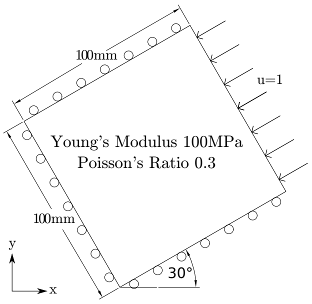

Figure 9: 30° Model

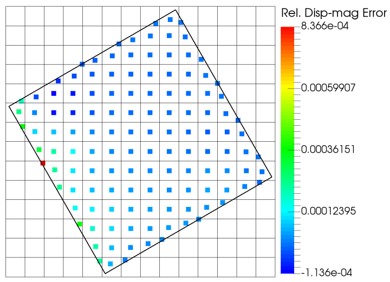

Figure 10: Relative Error in Displacement - 30° Model

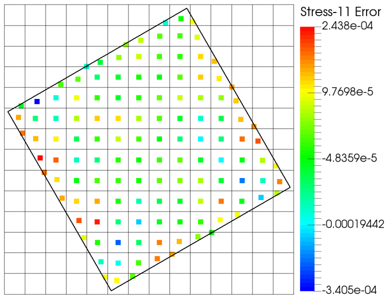

Figure 11: Relative Error in σ_1 - 30° Model

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