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Novel Finite Elements for initial value problems of light waves in the time domain

***M. Drolia[1], M.S.Mohamed[1], O. Laghrouche[1], M. Seaid[2], and J. Trevelyan[2]**

[1]Institue for Infrastructure and Environment, Heriot Watt University, Edinburgh EH14 4AS
[2]School of Engineering and Computing Sciences, Durham University, Durham DH1 3LE

*md272@hw.ac.uk

ABSTRACT

This paper proposes a novel scheme for the solution of Maxwell equations in the time domain. A discretization scheme in time is developed to render implicit solution of system of equations possible. The scheme allows for calculation of the field values at different time slices in an iterative fashion. This facilitates us to tackle problems whose solutions have harmonic or even more general dependency on time.
The spatial grid is partitioned into finite number of elements with intrinsic shape functions to form the bases of solution. Furthermore, the finite elements are enriched with plane wave functions. This significantly reduces the number of nodes required to discretize the geometry, without compromising on the accuracy or allowed tolerance in the errors, as compared to that of classical FEM. Also, this considerably reduces the computational costs, viz. memory and processing time. Parametric studies, presented herewith, confirm the robustness and efficiency of the proposed method.
The numerical scheme can thus be further developed for solution of problems where analytical solutions cannot be developed, or even when the solution cannot be categorized as time-harmonic in nature.

Key Words: *Finite Element ; Partition of Unity ; Time domain ; Wave Equations*

1. Transverse Electric Mode of propagation

Let Ω be a unit square defined on a 2D Euclidean space, with its four edges as the boundary Γ. The boundary value problem be defined as follows

$$\frac{\partial^2 E}{\partial t^2} - c^2\nabla^2 E = f(t,x,y); \qquad \text{on } \Omega \tag{1a}$$

$$\frac{\partial E}{\partial \hat{\mathbf{v}}} + hE = g(t,x,y); \qquad \text{on } \Gamma \tag{1b}$$

$$E^0 = U_0 \tag{1c}$$

$$\frac{\partial E^0}{\partial t} = V_0 \tag{1d}$$

where E is the magnitude of the transverse electric field in the direction $\hat{\mathbf{z}}$ perpendicular to the Euclidean plane. The above equation can be approximated using finite element and finite difference schemes for numerical solution. Let's discretize the time derivative in the following way (in order to facilitate the development of time-dependent formulations [1] as seen in analyses for transient response [2] or diffusion problems [3])

$$\frac{\partial^2 E^n}{\partial t^2} = \frac{E^n - 2E^{n-1} + E^{n-2}}{\Delta t^2} \tag{2}$$

Where the superscript n stands for the value of the field at the time instance $t = n\Delta t$. Substituting in (1a) gives

$$\nabla^2 E^n = \frac{E^n - 2E^{n-1} + E^{n-2}}{c^2\Delta t^2} - \frac{1}{c^2} f(t,x,y)$$

$$\Rightarrow E^n - (c^2\Delta t^2)\nabla^2 E^n = 2E^{n-1} - E^{n-2} + (\Delta t^2) f(t,x,y) \tag{3}$$

The equation (3) can be used to obtain a weak form which can be further solved over a finite number of elements in space as a linear system of equations. Let u be a test function multiplied to (3)

$$u(E^n - (c^2\Delta t^2)\nabla^2 E^n) = u\left(2E^{n-1} - E^{n-2} + (\Delta t^2)f(t,x,y)\right) \tag{4}$$

Integrating the left and right hand sides, over the domain Ω with boundary Γ, and applying the divergence theorem we get

$$\int_\Omega uE^n d\Omega + (c^2\Delta t^2)\left\{\int_\Omega \nabla u \cdot \nabla E^n d\Omega - \int_\Gamma u\,\hat{\mathbf{v}} \cdot \nabla E^n d\Gamma\right\} = \int_\Omega u\left(2E^{n-1} - E^{n-2} + (\Delta t^2)f(t,x,y)\right) d\Omega \tag{5}$$

Where $\hat{\mathbf{v}}$ is the normal unit vector to Γ. From (1b) and (5) we get the solvable weak form

$$\begin{aligned}\int_\Omega uE^n d\Omega \; + \; (c^2\Delta t^2)\int_\Omega \nabla u \cdot \nabla E^n d\Omega + (c^2\Delta t^2)\int_\Gamma u\,(hE^n) d\Gamma = \\ \int_\Omega u\left(2E^{n-1} - E^{n-2} + (\Delta t^2)f(t,x,y)\right) d\Omega + (c^2\Delta t^2)\int_\Gamma u\, g(t_n,x,y)) d\Gamma\end{aligned} \tag{6}$$

This equation (6) can be used to solve for E^n for the given set of boundary and initial conditions. The equation can then be iterated over n to obtain subsequent values of the fields for consecutive time steps.

This paper validates the proposed method against a transient wave problem on a 2D plane, where the solution is such that the magnitude of Electric field E is defined as $E = Ae^{\mathrm{i}wf_L^p}$ where $p = t - \frac{rk}{\omega}$. Here k is the wave number, ω angular frequency, r length of the position vector, e the natural exponent and i the imaginary number. Then the above constant c becomes the phase velocity defined such as $c = \frac{\omega}{k}$ while the function $g(t,x,y)$ is defined on each domain edge according to the relevant normal direction. The propagator function f_L^p (defined in the appendix) provides a means to control the initial condition of the problem, and can be used to manipulate the envelope of the moving wave, such that the solution is a wave expanding symmetrically about the origin as it evolves in time.
The problem is initialized with the solution E^0 (i.e. at $t = 0$), and for the boundaries we use appropriate derivatives.
To solve the weak form (6) using the finite element method we mesh the domain into a set of elements where the field E over each element is approximated in terms of a set of nodal values E_i and nodal shape functions N_i such as

$$E = \sum_{i=1}^{n} E_i N_i \tag{7}$$

Using the partition of unity [4] property one may further express the nodal values of the potential E_i as a combination of Q plane waves [5] such that

$$E = \sum_{i=1}^{n} N_i\left(B + \sum_{q=1}^{Q} A_i^q e^{\mathrm{i}(kx\cos\alpha_q + ky\sin\alpha_q)}\right) \tag{8}$$

where α_q is the angle of the qth plane wave. This ensures that we have the B term to capture variations which vary slowly (or don't vary at all, for example constants), and the plane wave enrichments that could form the basis for wave type solutions in the computational domain. Now by solving the linear system resulting from the above discrete representation we get the amplitudes A_i^q of the plane waves which is the qth plane wave contribution at the node i.

2. Analyses

A comparison of relative errors is conducted to study the behaviour of accuracy in solution obtained from the suggested PUFEM and classical FEM (figure 1). We test the accuracy of our method with the L_1 norm, computed as the relative error percentage given by $L_1 = \frac{\mathrm{abs}(\tilde{E}-E)}{\mathrm{abs}(E)} \times 100$, where $\tilde{E}, E$ are numerical and analytical solutions of the problem at hand.

The problems parameters are wave number $k = 8\pi$, angular frequency $\omega = 1$, and amplitude $A = 1$. The computational domain is a 2D unit square, with its bottom-left vertex shifted from the origin by a distance of $c \times 100\Delta t$ in each direction. This facilitates the wave, originating at the origin, to enter the computational domain by a 100 iterations in time, for the given step-size Δt. Note, the parameters assumed here are strictly numerical.

Table 1: : Parameters for PUFEM vs. FEM

Type	Δt	λ	DOF	Q	τ
PUFEM	10^{-2}	1/4	300	12	4.3
FEM	10^{-2}	1/4	3600	n/a	15

Table 1 shows the values of parameters studied. τ is the total number of degrees of freedom per wavelength, computed as $\tau = \lambda m\sqrt{Q}$ for PUFEM, and for FEM was calculated as $\tau = \lambda m$, where m is the number of nodes per direction on the computational grid. The total number of degrees of freedom (DOF) for PUFEM was computed as $m^2 Q$ and for FEM it's simply m^2.
Figure 1 below shows the plots of errors obtained from the analyses. The FEM stands at 50% error at the end of 5000 time steps (for a total time of 50 with $\Delta t = 10^{-2}$), as compared to the proposed PUFEM which showed less than 10% error with about one fourth the τ used in case of FEM.

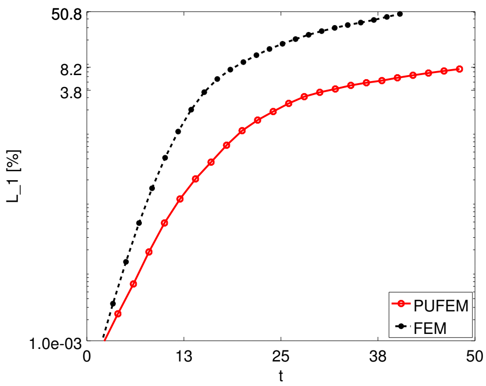
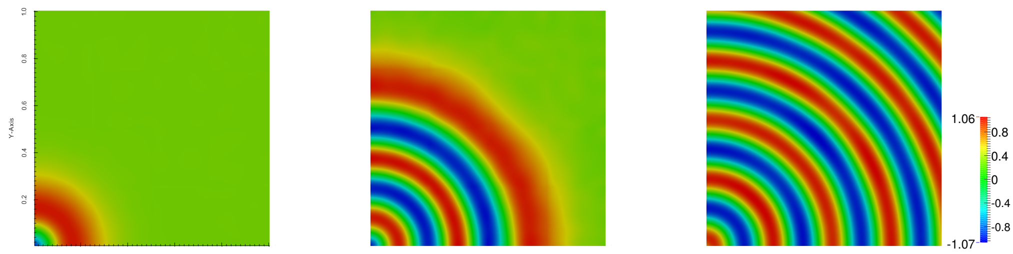


Figure 1: Semilog plot of L_1 norms in percentage to compare results from PUFEM and FEM for $k = 8\pi$. The wave covers the whole of computational domain by time $t \approx 36$.



(a) $t = 14$ (b) $t = 27$ (c) $t = 50$

Figure 2: 2D Plots of the recovered wave ($k = 8\pi$), obtained using the proposed PUFEM. The computational domain was meshed into 4×4 elements. The number of plane wave enrichments used $Q(\tau) = 12(4.3)$, and $\Delta t = 10^{-3}$. The final L_1 norm percentage at the end of the simulation was 3.6%

Even though we know that, in theory, a smaller Δt would lead to better results, however, it becomes

increasingly impractical to use smaller Δt with FEM owing to the sheer computational costs to solve bigger systems. To provide an idea (and by no means a rigourous comparison), the FEM results presented here took multiple weeks to compute, compared to their PUFEM counter parts which finished all the computations over a few hours.
Figure 2 shows 2D plots for numerical solution obtained with PUFEM when we used the same $\tau = 4.3$ but with a smaller $\Delta t = 10^{-3}$, and the final error was about 4%.

3. Conclusions

An enriched Finite Element Method, utilising the property of partition of unity to enrich the nodal values in the classical FEM, is formulated for solution of Maxwell equations in the time domain. The proposed PUFEM is validated against a progressive wave problem as demonstrated in section 2, wherein the method is tested against analytical solution for the proposed problem, with the L_1 norm. A comparison of the suggested method with classical FEM is carried, and it is observed that the former outperforms the latter on the grounds of lesser computational cost (including the total simulation time) and accuracy.

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Appendix A. Definition for the propagator function

The function definition for f_L for parameters a and b is given as

$$f_L(x) = \frac{1}{1+a}\left\{\operatorname{erf}\left(\frac{x}{b}\right)x + ax + \frac{b}{\sqrt{\pi}}e^{-\frac{x^2}{b^2}}\right\} \tag{A.1}$$

The parameters a and b can be set to control the smoothness of the slope of the function near the origin. The derivative of this function is given by

$$\frac{d}{dx}\left\{f_L(x)\right\} = \frac{\operatorname{erf}\left(\frac{x}{b}\right) + a}{1+a} \tag{A.2}$$

That is, the derivative of the function is a shifted and normalised error funciton. The following notation is used for the function

$$f_L(p) = \frac{1}{1+a}\left\{\operatorname{erf}\left(\frac{p}{b}\right)p + ap + \frac{b}{\sqrt{\pi}}e^{-\frac{p^2}{b^2}}\right\} = f_L^p \tag{A.3}$$