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Solving short wave problems using high order finite elements

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ABSTRACT

In this paper, we study numerical solutions of Helmholtz problems using high order finite element formulations. These include the polynomial high-order finite element method (high-order FEM) and the partition of unity finite element method (PUFEM). The aim of the study is to solve efficiently and with high level of accuracy two dimensional problems at high frequencies. The performance of both methods is compared and analysed on test examples of practical interest.

Key Words: *Helmholtz equation ; weak variational formulation ; partition of unity finite element method ; high order finite element method ; propagation and evanescent waves*

1. Introduction

Numerical solutions to the Helmholtz problems depend significantly on the wave number k. Specifically, the accuracy of the solution of the Helmholtz equation using the conventional FEM deteriorates with increasing wave number, even if the number of elements per wavelength is kept constant. This is due to the *pollution* error [1].

With the aim to reduce the computational cost and improve the accuracy of wave problems, various methods based on field enrichment have been proposed. We addressed the literature on partition of unity finite element method (PUFEM) and other related methods. Another alternative to reduce the pollution error is to increase the interpolation polynomial order over each element. High order polynomial approximations reduce resolution requirements which results in lower number of parameters in the whole problem.

The current work assesses both PUFEM and high-order FEM for the solution of Helmholtz problems with increasing wave numbers. PUFEM has been thoroughly investigated for acoustic [2] wave problems and attempts have been made to compare its performance to that of the standard FEM. Given that only low order elements have been considered for FEM, it is intended here to increase the polynomial order p to hopefully claim a fair comparison.

2. Weak formulation and numerical approximation

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with a smooth boundary Γ. We consider finite element discretizations for approximating the solution of the Helmholtz equation

$$-\Delta u - k^2 u = 0 \qquad \text{in} \quad \Omega, \tag{1a}$$

$$\nabla u.\mathbf{n} + \mathrm{i}ku = g \qquad \text{on} \quad \Gamma. \tag{1b}$$

The Galerkin finite element approximation is applied to the weak variational formulation of the Helmholtz problem. The weak formulation is obtained by multiplying the Helmholtz equation (1a) by a smooth test function $v = v(x, y)$ and integrating the resulting equation over the domain Ω such as

$$-\int_{\Omega} (\Delta u + k^2 u) v \, d\Omega = 0. \tag{2}$$

By applying the integration by parts to the integrand with second order derivatives and then introducing the Robin boundary condition (1b), the system of equations to solve is then

$$\int_{\Omega} (\nabla u.\nabla v - k^2 uv)\, d\Omega + \mathrm{i}k \oint_{\Gamma} u.v\, d\Gamma = \oint_{\Gamma} gv\, d\Gamma. \tag{3}$$

Our aim is to find an approximate solution u_h of the weak form (3) using either, high-order finite elements (FEM) or elements with plane wave enrichment functions (PUFEM).

In the high-order FEM model, the computational domain Ω is divided into N uniform non-overlapping quadrilateral elements Ω_e, $e = 1, ..., N$. The field unknown variable over each element Ω_e is then approximated by

$$u_h = \sum_{j=1}^{\#\text{vert}} u_j N_j(\boldsymbol{\xi}) \tag{4}$$

where N_j stands for the Lagrangian polynomial interpolation functions [3] and u_j represents the nodal values (unknowns of the problem) corresponding to the element vertices. The degree p of the polynomial interpolation functions N_j depends on the number of nodes assigned to each element. In general, for an approximation of degree p the number of vertices per element would be $(p+1)^2$.

In the PUFEM model, the elements are chosen to be quadrilateral with #vert = 4. At each vertex, the unknown variable u_j of expression (4) is expanded into a linear combination of q plane waves with directions encompassing the two dimensional space [2]. The PUFEM approximation of the unknown field variable within each element Ω_e is then given by

$$u_h = \sum_{j=1}^{4} \sum_{l=1}^{q} N_j\, e^{\mathrm{i}k\boldsymbol{d}.\boldsymbol{r}} A_j^l. \tag{5}$$

where N_j are the bi-linear basis functions corresponding to the mesh vertex, $\boldsymbol{d} = (\cos\theta_l, \sin\theta_l)^T$ and $\theta_l = 2\pi\ l/q$ for $l = 1, 2, ..., q$. The unknowns of the problem are no more the coefficients u_j but the amplitude factors A_j^l of the plane waves.

3. Numerical experiments

In this section, both PUFEM and high-order FEM models are assessed by considering test cases of practical interest. The first one involves an evanescent wave and the second case deals with the propagation of waves in a duct with rigid walls, which involves propagating and decaying modes. The performance of each method is measured through the relative error, in percent, using the L_2 norm. The discretization level in terms of degrees of freedom per wavelength is indicated by the paramter τ given by $\tau = \lambda \sqrt{totdof/area(\Omega)}$, where $totdof$ stands for the total number of degrees of freedom required for the solution and Ω_{area} is the area of the computational domain. Another parameter of interest is $totsys$ representing the total number of storage locations of the system matrix (x). Finally, the conditioning of the system matrix (x), denoted by $\kappa(A)$, is also considered. It is computed using the 1-norm and is given by $\kappa(A) = ||A||_1 ||A^{-1}||_1$. To conclude with, all the elementary integrals are evaluated numerically in a straight forward way by using Gauss-Legendre quadrature. The number of integration points is chosen to be enough so that the results are not affected by the integration errors.

3.1. Evanescent wave problem

The first test case deals with the numerical solution of an evanescent wave problem in a square domain $\Omega = [-1, 1] \times [-1, 1]$. The Robin boundary condition is applied on the boundary Γ through the boundary source term g. The exact solution of this problem can be found in [4].

The problem is solved for the wave number $ka = 100$ and $\beta = 1.001$. The results obtained with both approaches are given in Table 1. We list the L_2-error, the condition number κ and the total number

Table 1: Evanescent wave test case for $ka = 100$ and $\beta = 1.001$. Relative percentage L_2-error, $\epsilon_2[\%]$, conditioning, $\kappa(A)$, and total number of storage locations, $totsys$for different numbers of τ.

	p-FEM			PUFEM
	p=10	p=20	p=30	
$\epsilon_2[\%]$	3.44969 (4.5)	4.44105 (3.8)	1.55543 (3.8)	1.61219 (1.5)
	0.00053 (10.1)	0.00061 (6.4)	0.00025 (5.7)	0.08167 (1.6)
	0.00015(11.4)	0.00016 (7.6)	0.00023 (7.6)	0.00081 (1.7)
	0.00002 (13.9)	0.00002 (11.4)	0.00009 (9.5)	0.00080 (1.8)
$\kappa(A)$	0.80E+04 (4.5)	0.13E+05 (3.8)	0.23E+05 (3.8)	0.14E+09 (1.5)
	0.40E+05 (10.1)	0.38E+05 (6.4)	0.47E+05 (5.7)	0.97E+09 (1.6)
	0.54E+05 (11.4)	0.53E+05 (7.6)	0.85E+05 (7.6)	0.37E+10 (1.7)
	0.84E+05 (13.9)	0.12E+06 (11.4)	0.11E+06 (9.5)	0.5IE+14 (1.8)
$totsys$	1,973,161 (4.5)	2,386,381 (3.8)	3,520,981 (3.8)	102,870 (1.5)
	22,978,081 (10.1)	10,827,301 (6.4)	11,687,221 (5.7)	139,965 (1.6)
	32,645,341 (11.4)	18,614,761 (7.6)	27,471,961 (7.7)	182,760 (1.7)
	59,413,861 (13.9)	62,293,141 (11.4)	53,386,201 (9.5)	231,255 (1.8)

of storage locations of the final system matrix, $totsys$. The discretization level τ is also given and is presented between brackets.

The results from Table 1 show different aspects of the high-order FEM and PUFEM. As the order p increases in the FEM, the discretization level τ required to achieve a prescribed accuracy is decreased. But PUFEM seems to provide similar quality results for significantly lower values of the discretization level τ. For example, PUFEM provides 1.5% error with $\tau = 1.5$ whereas a similar error is obtained with $p = 30$ and $\tau = 3.8$. Moreover, PUFEM provides an error of order $10^{-4}\%$ with $\tau = 1.7$ whereas an error of the same order is achieved with $\tau = 10.1$ for $p = 10$, with $\tau = 6.4$ for $p = 20$ and finally, with $\tau = 5.7$ for $p = 30$. On the other hand, the lowest levels of L_2-error are achieved by the polynomial approach approach (FEM) but with a significantly higher discretization level. Moreover, the number $totsys$ of storage locations required by PUFEM is lower in comparison to the values required by high-order FEM. However, looking at the condition number κ, the values corresponding to high-order FEM are of the same order while the value corresponding to PUFEM is orders of magnitude higher.

3.2. Wave propagation in a duct

The second test example deals with the propagation of a wave in a duct with rigid walls. The computational domain $\Omega = [0,2] \times [0,1]$ is considered with the Robin condition (1b) on its boundary Γ through the source term g. The exact solution of this problem can be found in [5].

We solve the problem for the wave number $ka = 40$. Two different values of θ are considered which give the highest-propagating mode ($\theta = 12$) and the lowest-evanescent mode ($\theta = 13$). The computed L_2-errors, in percent, and the discretization levels are given in Table 2.

As shown in Table 2, the error decreases by refining the mesh grid for high-order FEM with a given order p and by increasing the number q of enriching plane waves for PUFEM. This is valid for both values of θ representing propagating and evanescent modes. Again, for all cases, PUFEM requires less degrees of freedom per wavelength in comparison to high-order FEM in order to reach a prescribed accuracy. While high-order FEM requires more degrees of freedom per wavelength, this number decreases as p increases. The results also show that the solution requires more degrees of freedom per wavelength to reach a certain accuracy for the evanescent wave problem in comparison to the propagating mode

Table 2: Wave propagation is a duct. Relative L_2-error [%] and discretization level for high-order FEM and PUFEM for $ka = 40$.

	p-FEM			PUFEM
	p=10	p=20	p=30	
$\theta = 12$	0.00380 (5.7)	0.00020 (4.6)	0.00101 (3.4)	1.24777 (1.9)
	0.00005 (9.0)	0.00006 (6.8)	0.00009 (6.8)	0.08853 (2.4)
	0.00005 (11.2)	0.00005 (11.2)	0.00007 (10.1)	0.00557 (2.7)
	0.00005 (13.4)	0.00006 (13.4)	0.00005 (13.4)	0.00019 (3.0)
$\theta = 13$	0.00870 (5.7)	0.00085 (4.6)	0.02018 (3.4)	39.14330 (1.9)
	0.00011 (9.0)	0.00010 (6.8)	0.00012 (6.8)	4.14675 (2.4)
	0.00010 (11.2)	0.00010 (11.2)	0.00010 (10.1)	0.41524 (2.7)
	0.00009 (13.4)	0.00010 (13.4)	0.00010 (13.4)	0.00121 (3.0)

problem. For example, PUFEM provides about 1.2% and 0.09% with $\tau = 1.9$ and 2.4, respectively, in the case of propagating mode. These errors increased to about 39.1% and 4.1% respectively in the case of evanescent mode. This observation is also valid for the high-order FEM approach.

4. Conclusions

In this work, two high order finite element formulations have been used to solve 2D Helmholtz problems. The methods chosen are the high-order polynomial finite element method (high-order FEM) and the partition of unity finite element method (PUFEM). The performance of each approach is assessed in terms of results quality and required degrees of freedom per wavelength. The condition number and total number of required storage locations are also considered.

PUFEM provides high quality results with a low number of degrees of freedom per wavelength, especially for relatively high frequencies where the element size incorporates many wavelengths. Errors lower than 1% were obtained with less than 2 degrees of freedom per wavelength. In such cases, the final system to solve is obviously drastically reduced in comparison to high-order FEM and hence the number of storage locations is also reduced. However, it is also shown that further increasing the discretization level by increasing the number of enriching plane waves does not always enhance the results beyond a certain level due to the ill-conditioning issue which is inherent to the plane wave enrichment technique.

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