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AN ITERATIVE LOCALLY CONSERVATIVE GALERKIN (LCG) METHOD FOR STUDYING FLOW IN A HUMAN ARTERIAL NETWORK

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ABSTRACT

In this study a robust iterative technique is developed for solving the one-dimensional human arterial blood flow problem by adopting Locally Conservative Galerkin method (LCG). Using Newton method with two different linear solvers (i.e. Gauss elimination and Jacobi methodologies), the non-linear governing equations are solved. Such strategies result in rapid convergence and fast solution without excessive memory cost for semi and full implicit LCG discretizations. In the proposed methods, the numerical strategies require computing a 4 x 4 matrix per element to determine blood flow characteristics. The novel methods developed are employed to study the blood flow through the major vessels of a complex human systemic circulation network. The validity and stability of the present methods are investigated by comparing the results against the available data in the literature.

Key Words: *Human Arterial Tree; One-dimensional Model; Locally Conservative Galerkin LCG; Finite Element Method; Newton-Gauss elimination and Newton-Jacobi.*

1. Introduction

Fundamental understanding of the blood flow behaviour is essential for predicting and treating common diseases in cardiovascular system such as aneurysm and atherosclerosis [1]. In order to achieve such comprehension in a complex geometry of human circulation, one-dimensional models have been used in most studies as they have been proven as efficient tools to give insight into quantities to interest [2, 3]. Among the schemes for solving the governing equations, explicit methodology in which the characteristic variables are employed to prescribe the boundary conditions is popular [2, 3]. Although such methods are more intuitive as the wave nature is considered and computationally efficient, time step restriction is the main drawback. Also, it leads to less stability especially at branching sites [4]. Alternatively, implicit treatment relaxes time step restriction but it is difficult to implement, especially in large scale problems due to large, complex and unsymmetric matrices which may lead to convergence issues. As the boundary conditions have to be prescribed in advance to allow a large time step, standard implicit schemes may affect the accuracy. Thus a combination of the advantages of steps from established explicit and implicit methods may lead to a better method. In the present work two such approaches , semi- and full- implicit Locally Conservative Galerkin (LCG) [5, 6] methods, are developed to relax the explicit LCG time step restriction and to enhance the stability. Basically, LCG method treats each element independently, this produces very small matrix of 4 x 4 per element and thus the solution is rapid. The standard Newton iteration is implemented here alongside a linear solver which is either Gauss elimination or Jacobi for the simultaneous solution. Such basic solvers are sufficient to achieve the rapid convergence since the matrices of LCG method are small unlike standard implicit methods. The results produced are compared against explicit methods to evaluate performance.

2. Governing Equations and Numerical Formulations

The governing equation (i.e. the conservation of mass and momentum equations) are numerically solved over 63 segments representing the major arteries in human systemic circulation as shown in Figure 1, also coronaries and ventricle valve are incorporated (All segments details can be found in [2, 3]). The compact form for the governing equation can be written as

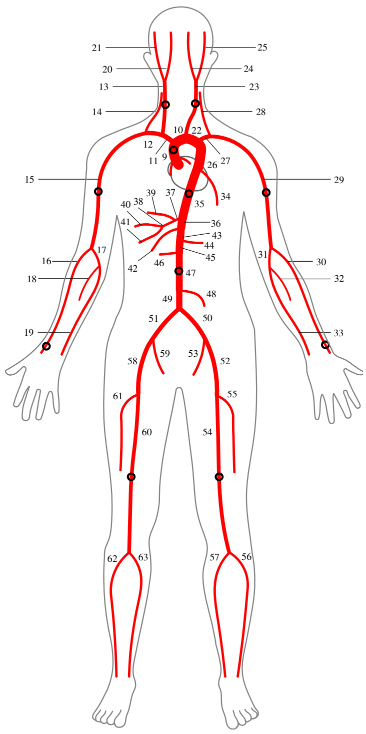

Figure 1: Configuration of the arterial tree for the current model [3].

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} - \mathbf{S} = 0 \tag{1}$$

where

$$\mathbf{U}=\begin{bmatrix} A \\ u \end{bmatrix}, \ \mathbf{F}=\begin{bmatrix} Au \\ \frac{u^2}{2} + \frac{p}{\rho} \end{bmatrix} \text{ and } \mathbf{S}=\begin{bmatrix} 0 \\ -\frac{8\pi\mu}{\rho}\frac{u}{A} \end{bmatrix}$$

In above equation, A and u are the primitive variables (i.e. area and velocity respectively). The pressure p is linked to the cross sectional area through the non-linear relationship, $p = p_{ext} + \beta(\sqrt{A} - \sqrt{A_o})$. The parameters, ρ, μ, p_{ext} and A_o are fluid density, fluid viscosity, the pressure from the surrounding tissues and the area at zero transmural pressure respectively. And, β accounts for the material properties of the elastic vessel and given as $\beta = \frac{\sqrt{\pi}hE}{A_o(1-\sigma)}$, where h is the vessel wall thickness, E is Young's modulus and σ is the Poisson's ratio, assumed to be 0.5 (i.e. the vessel wall is incompressible) (see [2] for more details). The finite elements discretisation procedures are applied to the governing equation according to [7] by adopting linear shape functions, which results in fully discrete LCG form as

$$[\mathbf{M}]_e\{\Delta\ \mathbf{U}^n\} = \Delta t \left[[\mathbf{K}]\{\mathbf{F}^{n+1}\} + [\mathbf{L}]\{\mathbf{S}^{n+1}\} + \{\mathbf{f}_\Gamma\}^{n+\theta} \right]_e \tag{2}$$

where e refers to an element and all matrices details (i.e. $\mathbf{M}$,$\mathbf{K}$ and $\mathbf{L}$) can be found in [2]. The flux $\mathbf{f}_\Gamma$ is used to transfer information between elements. As stated before, we use both versions of LCG method, i.e. semi- implicit in which $\mathbf{f}_\Gamma^n = \mathbf{F}^n$ at $\theta = 0$ and fully-implicit in which $\mathbf{f}_\Gamma^{n+1} = \mathbf{F}^{n+1}$ at $\theta = 1$. In both cases, the flux is rewritten in terms of implicit area and velocity at n or $n + 1$ time level. After that, the primitive variables in Eq.2 are estimated according to Newton method [8, 9] along with Gaussian elimination or Jacobi for solving the simultaneous equations.

3. Results and Discussion

As stated before, the governing equations along with the boundary conditions are solved over the whole arterial tree. The input wave for the model is the pressure wave from left ventricle (see [2, 3] for more details). The pressure and flow rate are briefly discussed here as shown in Figure 2 at two monitoring positions along the aorta, i.e. ascending aorta (segment 9, denoted by A and B in the figure). The results are produced using Newton-Gauss elimination and semi- and full-implicit LCG method (legends Case 1 and Case 2 respectively). The Thoracic aorta II is also considered in Figure 2 (segment 35, denoted by C and D in the figure) in which Newton-Jacobi is adopted. We run three cardiac cycles where time step is chosen as $\Delta t = 0.0002s$, larger time step is possible but due to diffusive nature of Taylor-Galerkin method at larger time steps, this is not attempted here. For explicit LCG method used for comparison, time step was set in [3] as $\Delta t = 2 \text{ x } 10^{-5}s$. The figure clearly demonstrates that new methods proposed are close to the established explicit method. Between the two methods the semi-implicit method appears to be least accurate. This may be due to the flux being treated at n time level.

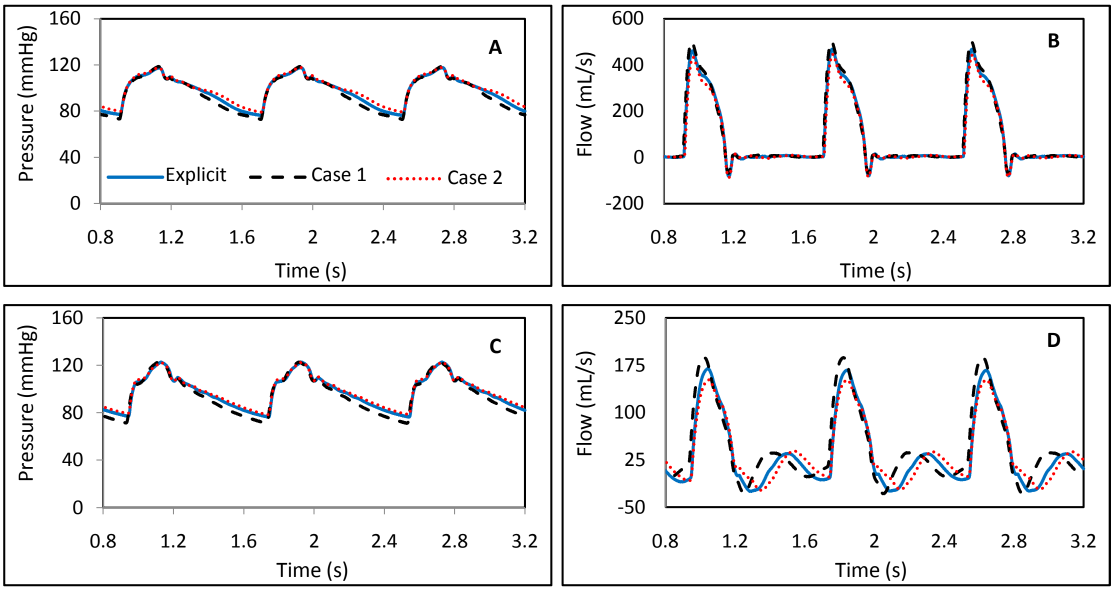

Figure 2: Comparisons for pressure and flow in various locations in the arterial tree.

In Table 1 the cost performance of the the three sachems are compared. As seen the fully-implicit method appears to be the fastest method among the methods compared and gives an accurate solution to the problem. The fully-implicit method also has the advantage of no time step restriction.

4. Conclusion

Two different time stepping schemes of LCG method, i.e. semi- and fully- implicit discretisations, have been developed for a human circulatory system. The solution is achieved by using the Newton iteration along with two linear solvers, Gauss elimination and Jacobi method. As LCG method generates only an element matrix (i.e. 4 x 4 size), this allows us to implement standard solvers. This enormously simplifies the computation. The comparisons between the proposed

Table 1: Computational speed comparison averaged over three cardiac cycles.

Method	Newton-Gauss, Semi-implicit	Newton-Gauss, Fully-implicit	Explicit
Time (s)	972.50	927.75	1017.00
Method	Newton-Jacobi, Semi-implicit	Newton-Jacobi, Fully-implicit	Explicit
Time (s)	942.00	896.25	1017.00

methods are established explicit method shows that the fully-implicit method is both fast and accurate. Thus the fully-implicit method proposed in this paper is recommended for future studies. Further investigation is essential to improve the fully-implicit method to enhance time step sizes, without compromising the accuracy.

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