Proceedings of the 24th UK Conference of the
Association for Computational Mechanics in Engineering
31 March - 1 April 2016, Cardiff University, Cardiff

A first order hyperbolic framework for large strain computational solid dynamics: A vertex-centred Updated Lagrangian scheme

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ABSTRACT

Current industrial codes (e.g. PAM-CRASH, ANSYS, LS-DYNA) for the simulation of large-scale solid dynamics are typically based on the use of traditional displacement-based finite element formulations. However, these formulations present a number of shortcomings, namely, (1) reduced order of convergence for strains and stresses, (2) poor performance in bending and shock dominated scenarios and (3) appearance of numerical instabilities associated with volumetric locking and pressure fluctuations.

To circumvent these drawbacks, a system of first order hyperbolic conservation laws for large strain solid dynamics was introduced [1–3]. Essentially, the formulation was established in terms of the linear momentum and the deformation gradient tensor [3]. This paper builds on recent Total Lagrangian based work [1] developed by the authors, by introducing a vertex centred Finite Volume Method on a mixed Updated Lagrangian formalism, with the aim of further enhancing the robustness of the algorithm. In the case of extremely high nonlinear deformations, the Updated Lagrangian framework provides extra flexibility into the formulation, thus extending the range of applications to near incompressibility. For computational efficiency, an adapted artificial compressibility approach is also introduced for truly and nearly incompressible materials [2].

A series of numerical examples are presented in order to assess the robustness and the applicability of the proposed framework, benchmarking it against alternative numerical strategies developed by the authors in recent publications [1–3]. The overall scheme shows excellent behaviour in shock and bending dominated nearly incompressible scenarios without spurious pressure oscillations, yielding second order of convergence for both velocities and stresses.

Key Words: *Fast solid dynamics ; Updated Lagrangian ; Finite volume ; Locking; Incompressibility*

1. Introduction

Over the last few years, the authors have introduced a new mixed Total Lagrangian conservation-based methodology, where the linear momentum $\boldsymbol{p}$ and the deformation gradient $\boldsymbol{F}$ are treated as primary variables of a system of first order conservation laws. Both velocities, deviatoric stresses and volumetric stresses display the same rate of convergence, which proves ideal in the case of linear finite elements. This approach has been thoroughly analysed by the authors using a wide variety of spatial second order discretisation techniques, namely cell centred upwind Finite Volume Method (FVM), vertex centred Jameson-Schmidt-Turkel (JST) and upwind FVM, two step Taylor Galerkin and Petrov Galerkin Finite Element Method (FEM). Moreover, the two-field $\boldsymbol{p}$-$\boldsymbol{F}$ formulation was then augmented by incorporating a new conservation law for the Jacobian of the deformation J to effectively solve nearly incompressible deformations. The $\boldsymbol{p}$-$\boldsymbol{F}$-J formulation was also extended to account for truly incompressible materials utilising a tailor-made fractional step Petrov Galerkin approach.

In this paper, the conservation-based $\boldsymbol{p}$-$\boldsymbol{F}$-J system through the use of either upwind or JST stabilisation is revisited, but this time established in terms of Updated Lagrangian formalism. This framework leads to a simpler expression for Jacobian evolution and, more importantly, further enhances the robustness of the algorithm. For nearly and truly incompressible materials, an adapted artificial compressibility approach is also introduced, with the aim of increasing computational efficiency.

2. Governing equations

The motion of a continuum, which occupies a volume Ω_0 in the reference configuration and a volume Ω in the deformed configuration, expressed in Updated Lagrangian formalism is described through the following ***p*-*F*-*J*** first order conservation laws:

$$\frac{d}{dt}\int_{\Omega_0} \boldsymbol{p}\, d\Omega_0 = \int_{\partial\Omega} \boldsymbol{t}\, da + \int_{\Omega} \boldsymbol{f}\, d\Omega; \tag{1a}$$

$$\frac{d}{dt}\int_{\Omega_0} \boldsymbol{F}\, d\Omega_0 = \int_{\partial\Omega} \left(\frac{\boldsymbol{p}}{\rho_0} \otimes \boldsymbol{H}^{-1}\right) \boldsymbol{n}\, da; \tag{1b}$$

$$\frac{d}{dt}\int_{\Omega_0} J\, d\Omega_0 = \int_{\partial\Omega} \frac{\boldsymbol{p}}{\rho_0} \cdot \boldsymbol{n}\, da; \tag{1c}$$

$$\frac{d}{dt}\int_{\Omega_0} E\, d\Omega_0 = \int_{\partial\Omega} \frac{\boldsymbol{p}}{\rho_0} \cdot \boldsymbol{t}\, da + \int_{\Omega} \frac{1}{\rho_0}\boldsymbol{f} \cdot \boldsymbol{p}\, d\Omega. \tag{1d}$$

where $\boldsymbol{p} := \rho_0 \boldsymbol{v}$ is the linear momentum, ρ_0 is the material density, $\boldsymbol{v}$ is the velocity field, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{f}$ is the body force per unit current volume, $\boldsymbol{F}$ is the deformation gradient (or fibre map), $\boldsymbol{H} := J\boldsymbol{F}^{-T}$ is the cofactor of the deformation gradient (or area map), J is the Jacobian of the deformation (or volume map), E is the total energy per unit undeformed volume and $\boldsymbol{t}$ represents the nominal traction related through the Cauchy stress tensor $\boldsymbol{\sigma}$ and the spatial unit outward normal vector $\boldsymbol{n}$. Crucially, the fibre map evolution (1b) needs to satisfy an additional set of compatibility conditions (also known as involutions), namely $CURL\boldsymbol{F} = \boldsymbol{0}$. In the context of isothermal process, the energy equation (1d) is redundant and is decoupled from the ***p*-*F*-*J*** system (1a-1c). However, equation (1d) can still be employed when measuring the amount of numerical dissipation introduced into the system.

For closure of the above system (1), appropriate initial and boundary conditions, as well as a constitutive model, are supplemented. For simplicity, a nearly incompressible neo-Hookean material is used by relating Cauchy stress $\boldsymbol{\sigma}$ with the geometrical measures $\{\boldsymbol{F}, J\}$:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{dev}(\boldsymbol{F}) + p(J)\boldsymbol{I}; \quad \boldsymbol{\sigma}_{dev} = \mu J_{\boldsymbol{F}}^{-5/3}\left[\boldsymbol{b} - \frac{1}{3}(\boldsymbol{F} : \boldsymbol{F})\boldsymbol{I}\right]; \quad p = \kappa(J-1), \tag{2}$$

where p is the pressure, $J_{\boldsymbol{F}} := \det \boldsymbol{F}$ is the Jacobian evaluated directly from the deformation gradient, $\boldsymbol{b} := \boldsymbol{F}\boldsymbol{F}^T$ is the left Cauchy-Green strain tensor, μ and κ are the shear and bulk moduli.

3. Finite volume spatial discretisation

From the spatial discretisation viewpoint, the above set of hyperbolic conservation laws (1) is semi-discretised using the vertex centred finite volume method widely accepted in fluid dynamics applications. The introduction of appropriate numerical viscosity is of paramount importance due to the hyperbolic nature of the system. In the following section, two variants of the numerical dissipative mechanisms are discussed, namely a nodally conservative JST stabilisation (see Section 3.1) and an Upwind Riemann solver stabilisation (see Section 3.2).

3.1. Jameson-Schmidt-Turkel (JST) nodally conservative stabilisation

Following Reference [3], one of the most effective options is to incorporate a nodally conservative numerical viscosity (e.g. second order and fourth order stabilisations) into the system without amending the flux computations. The fourth order term $\mathcal{D}_4$ (biharmonic) provides stabilisation against odd-even decoupling phenomena, whilst the second order term $\mathcal{D}_2$ (harmonic) is activated with steep gradients, capturing any discontinuity in the solution. For simplicity, we restrict ourselves to problems where no physical shocks occur. For this reason, there is no need to consider harmonic stabilisations in our system:

$$\Omega_0^a \frac{d\boldsymbol{p}_a}{dt} = \sum_{b\in\Lambda_a} \frac{1}{2}(\boldsymbol{\sigma}_a + \boldsymbol{\sigma}_b)\boldsymbol{c}_{ab} + \sum_{\gamma\in\Gamma_a^B} \boldsymbol{t}_a^\gamma \frac{a^\gamma}{3} + \mathcal{D}_4(\boldsymbol{p}_a); \tag{3a}$$

$$\Omega_0^a \frac{d\boldsymbol{F}_a}{dt} = \sum_{b\in\Lambda_a} \frac{1}{2\rho_0}(\boldsymbol{p}_a + \boldsymbol{p}_b) \otimes \boldsymbol{C}_{ab} + \sum_{\gamma\in\Gamma_a^B} \frac{1}{\rho_0}\left(\boldsymbol{p}_a^\gamma \otimes \boldsymbol{N}^\gamma\right)\frac{A^\gamma}{3}; \tag{3b}$$

$$\Omega_0^a \frac{dJ_a}{dt} = \sum_{b\in\Lambda_a} \frac{1}{\rho_0}\boldsymbol{p} \cdot \boldsymbol{c}_{ab} + \sum_{\gamma\in\Gamma_a^B} \frac{1}{\rho_0}\boldsymbol{p}_a^\gamma \cdot \boldsymbol{n}^\gamma \frac{a^\gamma}{3} + \mathcal{D}_4(J_a), \tag{3c}$$

where $\boldsymbol{C}_{ab}$ and $\boldsymbol{c}_{ab}$ are the initial and current area vectors and $\mathcal{D}_4(\boldsymbol{\beta}_a)$ represents the fourth order JST dissipation defined as:

$$\mathcal{D}_4(\boldsymbol{\beta}_a) = -k^{(4)} \sum_{b\in\Lambda_a} \Psi_{ab}(c_p)\theta_{ab}(\boldsymbol{L}(\boldsymbol{\beta}_b) - \boldsymbol{L}(\boldsymbol{\beta}_a)); \quad \boldsymbol{L}(\boldsymbol{\beta}_a) = \sum_{b\in\Lambda_a} \theta_{ab}(\boldsymbol{\beta}_b - \boldsymbol{\beta}_a); \quad \boldsymbol{\beta}_a = \{\boldsymbol{p}_a, J_a\}. \tag{4}$$

Here, $k^{(4)}$ is a user-defined parameter, $\Psi_{ab}(c_p)$ is the spectral radius depending only on the volumetric wave c_p and θ_{ab} is a geometrical weight used to preserve second order accuracy in unstructured meshes. For the satisfaction of involutions, it is crucial to ensure that no dissipation is introduced in the fibre map evolution (3b) by approximating the corresponding fluxes via central difference method [3].

3.2. Upwind stabilisation: Acoustic Riemann solver

In contrast to the JST stabilisations described above, a more refined approach by adding appropriate conservative numerical viscosity through fluxes is introduced [1]. Naturally, discontinuity of the conservation variables across control volume interfaces leads to a Riemann problem. A suitable numerical interface (acoustic) flux can be obtained by making use of the Rankine-Hugoniot jump conditions, of standard use within the field of fluid dynamics:

$$\Omega_0^a \frac{d\boldsymbol{p}_a}{dt} = \sum_{b\in\Lambda_a} \boldsymbol{t}^C \|\boldsymbol{c}_{ab}\| + \sum_{\gamma\in\Gamma_a^B} \boldsymbol{t}_a^\gamma \frac{a^\gamma}{3}; \tag{5a}$$

$$\Omega_0^a \frac{d\boldsymbol{F}_a}{dt} = \sum_{b\in\Lambda_a} \frac{1}{2\rho_0} (\boldsymbol{p}_a + \boldsymbol{p}_b) \otimes \boldsymbol{C}_{ab} + \sum_{\gamma\in\Gamma_a^B} \frac{1}{\rho_0} \left(\boldsymbol{p}_a^\gamma \otimes \boldsymbol{N}^\gamma\right) \frac{A^\gamma}{3}; \tag{5b}$$

$$\Omega_0^a \frac{dJ_a}{dt} = \sum_{b\in\Lambda_a} \frac{1}{\rho_0} \boldsymbol{p}^C \cdot \boldsymbol{c}_{ab} + \sum_{\gamma\in\Gamma_a^B} \frac{1}{\rho_0} \boldsymbol{p}_a^\gamma \cdot \boldsymbol{n}^\gamma \frac{a^\gamma}{3}, \tag{5c}$$

where

$$\boldsymbol{t}^C = \frac{1}{2}(\boldsymbol{\sigma}^- + \boldsymbol{\sigma}^+)\,\boldsymbol{n} + \frac{1}{2}\left[\frac{1}{c_p}(\boldsymbol{n}\otimes\boldsymbol{n})(\boldsymbol{\sigma}^+ - \boldsymbol{\sigma}^-)\boldsymbol{n} + \frac{1}{c_s}(\boldsymbol{I} - \boldsymbol{n}\otimes\boldsymbol{n})(\boldsymbol{\sigma}^+ - \boldsymbol{\sigma}^-)\boldsymbol{n}\right]; \tag{6a}$$

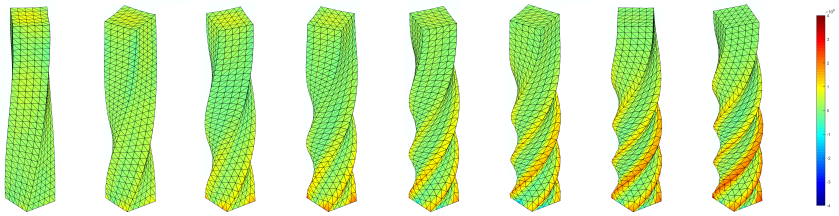
$$\boldsymbol{p}^C = \frac{1}{2}(\boldsymbol{p}^+ + \boldsymbol{p}^-) + \frac{1}{2c_p}(\boldsymbol{\sigma}^+ - \boldsymbol{\sigma}^-)\boldsymbol{n}; \quad \boldsymbol{\sigma}^{-,+} = \boldsymbol{\sigma}_{dev}^{-,+} + \kappa(J^{-,+} - 1)\boldsymbol{I}. \tag{6b}$$

Notice that this type of upwinding stabilisations (which depends on both volumetric wave c_p and shear wave c_s) is derived with the consideration of wave directional character, being more accurate in comparison to JST stabilisation [1].

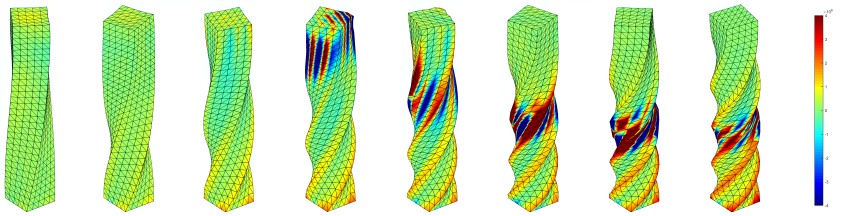
Finally, the set of stabilised semidiscrete equations (see (3) and (5)) can then be explicitly integrated from time step t^n to t^{n+1}. In this case, the explicit one-step two-stage Total Variation Diminishing Runge Kutta (TVD-RK) time integrator [2] is preferred due to its excellent TVD properties. The geometry is also advanced in time using the exact same time integrator in a monolithic manner. As described in [1, 3], the scheme is suitably modified to guarantee the conservation of angular momentum. In the case of nearly and truly incompressible materials, the volumetric wave speed c_p can reach very high values leading to prohibitively small time steps. This can have a very negative effect in the computational efficiency of the algorithm. Taking inspiration from Reference [2], a tailor-made artificial compressibility algorithm is also introduced for the three-field ***p*-*F*-*J*** mixed formulation.

4. Numerical examples

A column with a unit square cross section is subject to different loading conditions; a prescribed sinusoidal angular velocity (see Figure 1) and a linearly varying velocity along its height (see Figures 2 and 3). The examples show the applicability and robustness of the scheme in truly and nearly incompressible bending dominated scenarios.

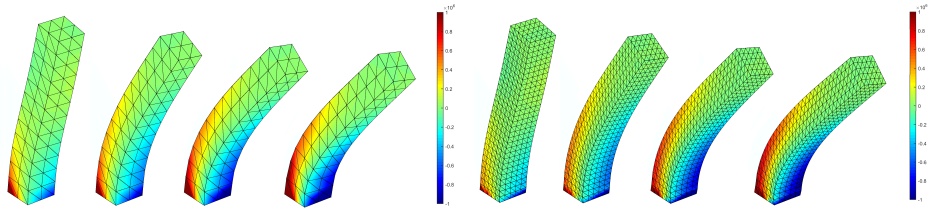


(a) Explicit Updated Lagrangian ***p*-*F*-*J*** Upwind algorithm



(b) Explicit Total Lagrangian ***p*-*F*-*J*** Upwind algorithm

Figure 1: Illustrate the robustness of Updated Lagrangian formalism over Total Lagrangian formalism.



(a) Explicit Updated Lagrangian ***p*-*F*-*J*** Upwind (b) Explicit Updated Lagrangian ***p*-*F*-*J*** JST

Figure 2: Demonstrate the accuracy of the stabilisations: Upwind is less dissipative than JST.

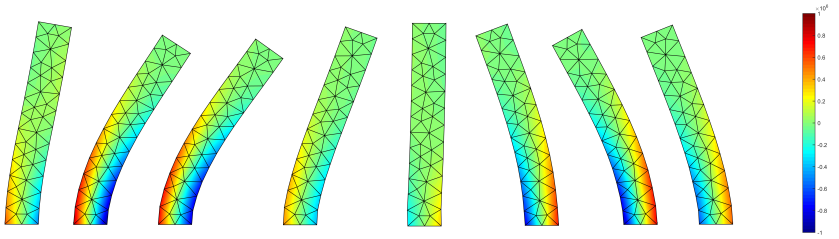


Figure 3: Robustness of ***p*-*F*-*J*** Upwind artificial compressibility in truly incompressible material.

5. Conclusion

This paper introduces a second order Updated Lagrangian vertex centred finite volume algorithm in the numerical simulation of large strain solid dynamics. A mixed formulation written in the form of a system of first order hyperbolic equations is employed. The linear momentum $\boldsymbol{p}$, the deformation gradient $\boldsymbol{F}$ and the Jacobian J are regarded as primary conservation variables. The overall scheme shows excellent behaviour in truly and nearly incompressible bending dominated scenarios without spurious pressure oscillations, yielding second order of convergence for velocities and stresses.

Acknowledgement

The first author gratefully acknowledge the financial support provided by the Erasmus Mundus Program (SEED). The second and third authors acknowledge the financial support provided by the Sêr Cymru National Research Network for Advanced Engineering and Materials, United Kingdom.

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