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A stabilised Total Lagrangian Corrected Smooth Particle Hydrodynamics (CSPH) in large strain explicit solid dynamics

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ABSTRACT

An explicit Total Lagrangian mixed momentum/strains formulation [2], in the form of a system of first order conservation laws, has been recently proposed to overcome the shortcomings posed by the traditional second order displacement-based formulation, namely: (1) bending and volumetric locking difficulties; (2) hydrostatic pressure fluctuations; and (3) reduced order of convergence for derived variables. Following the work of Bonet and Kulasegaram [4, 5], the main objective of this paper is the adaptation of Corrected Smooth Particle Hydrodynamics (CSPH) in the context of Total Lagrangian mixed formulation. Appropriate nodally conservative Jameson-Schmidt-Turkel (JST) stabilisation is introduced taking advantage of the conservation laws. A mixed linear momentum/deformation gradient technique performs extremely well in nearly incompressible bending dominated scenarios [1, 2] without the appearance of spurious pressure oscillations. As both linear momentum and deformation gradient are used as primary variables of the system, equal order of approximation is achieved in both fields. A series of numerical examples are carried out to assess the applicability and robustness of the proposed algorithm.

Keywords: *Conservation laws; SPH; Locking; Incompressibility; Fast dynamics; JST*

1. Introduction

Dynamic explicit displacement-based finite element codes, based on low order finite element technology, are commonly used for the simulation of large strain impact problems by aerospace, automotive, biomedical, defence and manufacturing industries. In these codes, the 8-noded underintegrated hexahedral element is the preferred option to model solid components. Many practical applications (i.e. crashworthiness and drop-impact modelling), however, experience extremely large solid deformations accompanied by severe mesh distortion. This may lead to poorly shaped elements unless some form of adaptive remeshing is applied.

From the viewpoint of spatial discretisation, the standard displacement-based formulation for low order elements is known to experience locking difficulties in nearly-incompressible, bending dominated scenarios. It is also known that the use of linear interpolation within a finite element leads to second order convergence for the primary variables, but one order less for derived variables. From the time discretisation point of view, the Newmark method has a tendency to introduce high frequency noise in the solution field, especially in the vicinity of sharp spatial gradients.

To overcome the shortcomings mentioned above, a mixed methodology [1, 2] is presented in the form of a system of first order conservation laws, where the linear momentum and the deformation gradient tensor are regarded as the two main conservation variables. In addition, this approach enables the deviatoric and volumetric stresses (or strains) to converge at the same rate as the velocities (or displacements).

The main goal of this paper is the introduction of a stabilised Corrected Smooth Particle Hydrodynamics (CSPH) methodology, tailor-made for this mixed formulation. The adoption of a system of first order conservation laws allows CFD techniques to be introduced for the very first time within the context of particle methods, in the form of an adapted nodally conservative JST stabilisation [2]. JST will be used to provide artificial (numerical) dissipation and shock-capturing capabilities. In particular, the second order (harmonic) and fourth order (biharmonic) operators employed in the JST stabilisation can be readily obtained by closed-form differentiation of the interpolating kernel functions. A quintic kernel approximation [4, 5] is suitably used, due to the presence of the biharmonic JST operator.

2. Reversible elastodynamics

The motion of a deformable continuum body from an initial (or undeformed) configuration $\boldsymbol{X} \in V \subset \mathbb{R}^3$ to a current (or deformed) configuration $\boldsymbol{x} \in v(t) \subset \mathbb{R}^3$ can be described by the relation $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{X}, t)$, where $\boldsymbol{\phi}$ links each material particle in $\boldsymbol{X}$ to its current position $\boldsymbol{x}$ at time t during the motion. In the case of isothermal process, a mixed formulation in the form of a system of first order conservation laws for elastodynamics can be cast as [1, 2]:

$$\frac{\partial \boldsymbol{p}}{\partial t} - \boldsymbol{\nabla}_0 \cdot \boldsymbol{P}(\boldsymbol{F}) = \rho_0 \boldsymbol{b}\,; \qquad \frac{\partial \boldsymbol{F}}{\partial t} - \boldsymbol{\nabla}_0 \cdot \left(\frac{1}{\rho_0} \boldsymbol{p} \otimes \boldsymbol{I}\right) = \boldsymbol{0}. \tag{1}$$

Here, $\boldsymbol{p} = \rho_0 \boldsymbol{v}$ is the linear momentum per unit of undeformed volume, ρ_0 is the material density, $\boldsymbol{v}$ represents the velocity field, $\boldsymbol{P}$ is the first Piola-Kirchhoff stress tensor, $\boldsymbol{b}$ is the body force per unit mass, $\boldsymbol{F}$ is the deformation gradient and $\boldsymbol{I}$ is the identity matrix. The notation $\boldsymbol{\nabla}_0$ denotes the material gradient operator in undeformed space, where $[\boldsymbol{\nabla}_0]_I \equiv \frac{\partial}{\partial X_I}$. In a more compact form, system (1) reads:

$$\frac{\partial \boldsymbol{\mathcal{U}}}{\partial t} + \frac{\partial \boldsymbol{\mathcal{F}}_I}{\partial X_I} = \boldsymbol{\mathcal{S}}; \qquad \forall I = 1, 2, 3; \tag{2}$$

where $\boldsymbol{\mathcal{U}}$ is the vector of conservation variables, $\boldsymbol{\mathcal{F}}_I$ is the flux vector in the material direction I and $\boldsymbol{\mathcal{S}}$ is the source term described by:

$$\boldsymbol{\mathcal{U}} = \begin{bmatrix} \boldsymbol{p} \\ \boldsymbol{F} \end{bmatrix}; \qquad \boldsymbol{\mathcal{F}}_N = \boldsymbol{\mathcal{F}}_I N_I = \begin{bmatrix} -\boldsymbol{P}\boldsymbol{N} \\ -\frac{1}{\rho_0}\, \boldsymbol{p} \otimes \boldsymbol{N} \end{bmatrix}; \qquad \boldsymbol{\mathcal{S}} = \begin{bmatrix} \rho_0 \boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}. \tag{3}$$

For closure of the above system (1), it is necessary to introduce an appropriate constitutive law by relating $\boldsymbol{P}$ with $\boldsymbol{F}$. For simplicity, a standard hyperelastic neo-Hookean model is used, whereby the first Piola $\boldsymbol{P}$ can be decomposed into the summation of deviatoric, $\boldsymbol{P}_{dev}$, and volumetric, $\boldsymbol{P}_{vol}$, contributions:

$$\boldsymbol{P} = \boldsymbol{P}_{dev} + \boldsymbol{P}_{vol}; \quad \boldsymbol{P}_{dev} = \mu J^{-2/3} \left(\boldsymbol{F} - \frac{1}{3}(\boldsymbol{F} : \boldsymbol{F})\, \boldsymbol{F}^{-T}\right); \quad \boldsymbol{P}_{vol} = p\boldsymbol{H}; \quad p = \kappa(J-1); \quad J = \det \boldsymbol{F}; \quad \boldsymbol{H} = J\boldsymbol{F}^{-T}. \tag{4}$$

3. Numerical methodology

In this section, the mixed $\boldsymbol{p}$-$\boldsymbol{F}$ system described by equation (1) will be discretised in space using the CSPH particle method with added dissipation from the JST technique (*$\boldsymbol{p}$-$\boldsymbol{F}$ JST-CSPH* scheme). To achieve this, and following the work of Bonet and co-authors [4, 5], the weak statement for the linear momentum evolution (1a) must be obtained through the use of work-conjugate principles [1] and integration by parts:

$$\int_V \delta \boldsymbol{v} \cdot \frac{\partial \boldsymbol{p}}{\partial t} dV = \int_V \delta \boldsymbol{v} \cdot \rho_0 \boldsymbol{b} dV + \int_{\partial V} \delta \boldsymbol{v} \cdot \boldsymbol{t}^B dA - \int_V \boldsymbol{P} : \boldsymbol{\nabla}_0 \delta \boldsymbol{v}\, dV. \tag{5}$$

Upon application of the particle integration on the above expression (5), and using the kernel approximation $\boldsymbol{\nabla}_0 \delta \boldsymbol{v}(\boldsymbol{X}) = \sum_{b \in \Lambda_X^b} V_b\, \delta \boldsymbol{v}_b \otimes \boldsymbol{\nabla}_0 W_b(\boldsymbol{X})$ to express the arbitrary virtual velocity field $\delta \boldsymbol{v}$, one obtains:

$$\frac{d\boldsymbol{p}_a}{dt} = \rho_0 \boldsymbol{b}_a + \frac{A_a}{V_a} \boldsymbol{t}_a^B + \sum_{b \in \Lambda_a^b} V_b\, (\boldsymbol{P}_b + \boldsymbol{P}_a)\, \tilde{\boldsymbol{\nabla}}_0 W_b(\boldsymbol{X}_a). \tag{6}$$

where V_b is the volume fraction assigned to the particle in $\boldsymbol{x}_b$ and $W_b(\boldsymbol{X})$ is the kernel function value there. Following References [3–6], the first Piola-Kirchhoff stress tensors $\{\boldsymbol{P}_a, \boldsymbol{P}_b\}$ are further approximated by using the coefficients directly from the particles. In (6), corrections are introduced on the kernel gradient to yield $\tilde{\boldsymbol{\nabla}}_0 W_b(\boldsymbol{X}_a)$, ensuring the correct evaluation of a gradient of a general linear function [4]. However, the above semi-discrete formulation still suffers from the appearance of accumulated numerical instabilities over a long term response. To put a remedy to this, in contrast to the dissipative mechanism introduced by Monaghan [4] through viscous fluxes, here an adapted nodally conservative JST stabilisation $\boldsymbol{\mathcal{D}}(\boldsymbol{p}_a)$ will be incorporated into (6), mirroring CFD techniques [2]:

$$\frac{d\boldsymbol{p}_a}{dt} = \rho_0 \boldsymbol{b}_a + \frac{A_a}{V_a} \boldsymbol{t}_a^B + \sum_{b \in \Lambda_a^b} V_b\, (\boldsymbol{P}_b + \boldsymbol{P}_a)\, \tilde{\boldsymbol{\nabla}}_0 W_b(\boldsymbol{X}_a) + \boldsymbol{\mathcal{D}}(\boldsymbol{p}_a). \tag{7}$$

The nodally conservative JST stabilisation is additively decomposed into a second order (harmonic) operator $\boldsymbol{\mathcal{D}}_2(\boldsymbol{p}_a)$ and a fourth order (biharmonic) operator $\boldsymbol{\mathcal{D}}_4(\boldsymbol{p}_a)$:

$$\boldsymbol{\mathcal{D}}(\boldsymbol{p}_a) = \boldsymbol{\mathcal{D}}_2(\boldsymbol{p}_a)+\boldsymbol{\mathcal{D}}_4(\boldsymbol{p}_a) = \kappa^{(2)} c_p \, h_{\min} \sum_{b\in\Lambda_a^b} V_b \, (\boldsymbol{p}_b - \boldsymbol{p}_a) \, \tilde{\Delta}_0 W_b(\boldsymbol{X}_a) - \kappa^{(4)} c_p \, h_{\min}^3 \sum_{b\in\Lambda_a^b} V_b \, (\Delta_0 \boldsymbol{p}_b - \Delta_0 \boldsymbol{p}_a) \, \tilde{\Delta}_0 W_b(\boldsymbol{X}_a), \tag{8}$$

where c_p is the pressure wave speed, h_{min} is the particle spacing, $\kappa^{(2)}$ and $\kappa^{(4)}$ are user-defined parameters and $\tilde{\Delta}_0$ represents the corrected Laplacian operator [6]. In regard to the fibre map evolution (1b), the application of particle collocation method [4] directly on (1b), along with the use of kernel approximation W, leads to:

$$\frac{d\boldsymbol{F}_a}{dt} = \boldsymbol{\nabla}_0 \boldsymbol{v}\big|_a \approx \sum_{b\in\Lambda_a^b} \frac{V_b}{\rho_0} \, (\boldsymbol{p}_b - \boldsymbol{p}_a) \otimes \tilde{\boldsymbol{\nabla}}_0 W_b(\boldsymbol{X}_a). \tag{9}$$

Finally, the set of stabilised particle equations (7) and (9) can then be explicitly integrated from time step t^n to t^{n+1}. In this case, the explicit one-step two-stage Total Variation Diminishing Runge Kutta (TVD-RK) time integrator [1, 2] is preferred due to its excellent TVD properties. Along with $\boldsymbol{p}$ and $\boldsymbol{F}$, the geometry, $\boldsymbol{x}$, is also advanced in time using the exact same time integrator in a monolithic manner. As described in [1], the scheme is suitably modified to guarantee the conservation of angular momentum.

4. Applications

A series of benchmark examples are assessed in order to test the applicability and effectiveness of the $\boldsymbol{p}$-$\boldsymbol{F}$ JST-CSPH mixed algorithm over the Total Lagrangian CSPH displacement based formulation [4, 5].

- Tension instability [5, 6]: a 1D bar is clamped on one end and left free at the other, where is given an initial velocity of $5\,m/s$. Fig. 1a shows the velocity history at the free tip. The bar is discretised in space using the $\boldsymbol{p}$-$\boldsymbol{F}$ JST-CSPH method. The JST term here has a decisive effect towards the achievement of a more accurate solution. The Total Lagrangian setting ensures the simulation does not suffer from tensile instability, a recurring issue for solid mechanics SPH simulations.

- Bending scenario [1, 2]: a bottom end clamp is imposed at $t = 0\,s$ on a 3D column of length $6\,m$, with unit cross section, initially travelling at constant side velocity $v = 10\,m/s$. Fig. 1b testifies to the good performance of the model in a bending dominated scenario.

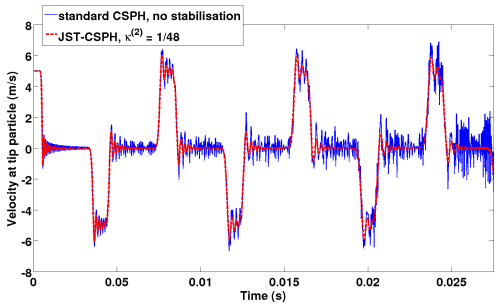
- Robustness [1, 2]: the column of the previous example is subjected to a sinusoidal angular velocity and its bottom is clamped. A comparison between fig. 1c (standard CSPH) and fig. 1d ($\boldsymbol{p}$-$\boldsymbol{F}$ JST-CSPH) illustrates the robustness of the mixed Total Lagrangian method [4, 5]: while the simulation in fig. 1c fails after a few time steps, the one in fig. 1d accurately captures the structure behaviour.

- Extremely large distortions [6]: a rectangular region of $3\,cm \times 1\,cm$ is punched with a constant vertical velocity of $-10\,m/s$ on one-third of the top surface. As reported in [6], the standard CSPH method [4, 5] is not capable of solving this problem, which involves extremely large distortions. This deficiency can be alleviated using the proposed algorithm (see fig. 1e).

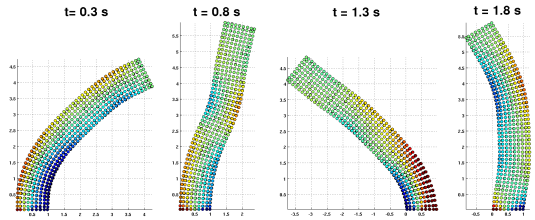
- Pressure instabilities [1, 2]: a 3D rubber cube of unit side length is left free at the top face and is constrained with rollers on the other sides. The right half of the domain experiences a prescribed punch velocity of $-100\,m/s$. As seen in fig. 1f, the $\boldsymbol{p}$-$\boldsymbol{F}$ JST-CSPH algorithm eliminates the appearance of non-physical pressure fluctuations in near incompressibility.

5. Conclusions

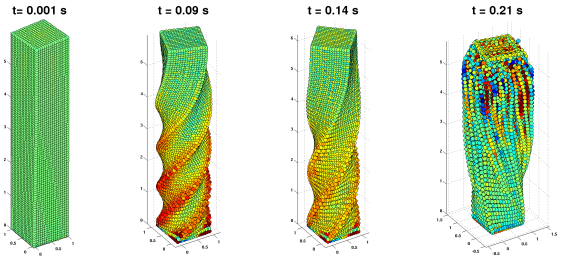
In this paper, a stabilised Total Lagrangian mixed formulation $\boldsymbol{p}$-$\boldsymbol{F}$ JST-CSPH algorithm is introduced for the numerical analysis of large strain solid dynamics. The methodology is based upon a system of first order conservation laws, where the linear momentum $\boldsymbol{p}$ and the deformation gradient $\boldsymbol{F}$ are regarded as the primary variables. A nodally conservative JST stabilisation is incorporated into the CSPH algorithm, taking advantange of the conservation laws. Finally, results from numerical simulations of benchmark problems are presented in order to demonstrate the applicability and robustness of the proposed methodology.



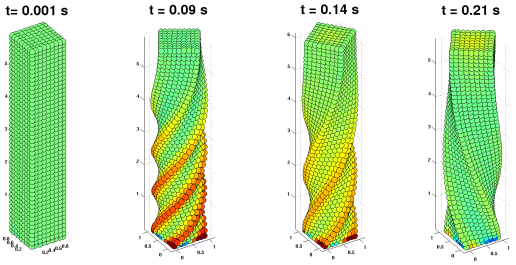
(a) 1d tensile test, ***p*-*F*** JST-CSPH based model



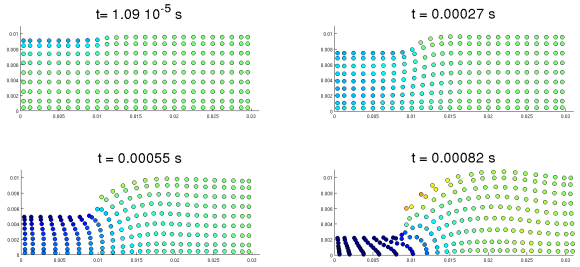
(b) Bending simulation, ***p*-*F*** JST-CSPH based model [1, 2]



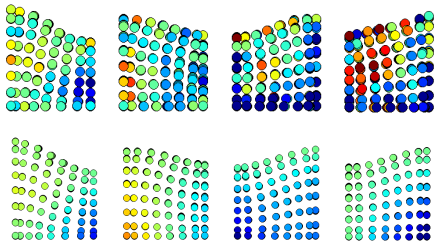
(c) Robustness: Standard CSPH based model [4, 5]



(d) Robustness: ***p*-*F*** JST-CSPH based model [1, 2]



(e) Large distortions: ***p*-*F*** JST-CSPH [1, 2]



(f) Pressure instabilities: [Top] Standard CSPH [4, 5]; [Bottom] ***p*-*F*** JST-CSPH [1, 2]

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References

[1] M. Aguirre, A. J. Gil, J. Bonet, A. A. Carreño. A vertex centred Finite Volume Jameson-Schmidt-Turkel algorithm for a mixed conservation formulation in solid dynamics. *JCP*, 259, 672–699, 2014.

[2] A. J. Gil, C. H. Lee, J. Bonet and R. Ortigosa. A first order hyperbolic framework for large strain computational solid dynamics. Part II: Total Lagrangian compressible, nearly incompressible and truly incompressible elasticity. *CMAME*, 300, 146–181, 2016.

[3] J. J. Monaghan. Smoothed particle hydrodynamics. *ARA&A*, 30, 543–574, 1992.

[4] J. Bonet, S. Kulasegaram. Correction and stabilization of smooth particle hydrodynamics methods with applications in metal forming simulations. *IJNME*, 47, 1189–1214, 2000.

[5] J. Bonet, S. Kulasegaram. Remarks on tension instability of Eulerian and Lagrangian Corrected Smooth Particle Hydrodynamics (CSPH) methods. *IJNME*, 52, 1203–1220, 2001.

[6] Y. Vidal, J. Bonet and A. Huerta. Stabilized updated Lagrangian corrected SPH for explicit dynamic problems. *IJNME*, 69, 2687–2710, 2007.