Topology optimisation using level set methods and the discontinuous Galerkin method

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ABSTRACT

This paper presents a topology optimisation approach that combines an adjoint-based sensitivity analysis [1] with level set methods (LSM) [2] for front propagation, and the discontinuous Galerkin (DG) symmetric interior penalty (SIP) method [3]. The problems considered in this paper will be limited to the minimum compliance design of two-dimensional linear elastic structures.

Key Words: *topology optimisation; level set methods; discontinuous Galerkin method; symmetric interior penalty method.*

1. Introduction

One of the most challenging aspects of structural design is finding the optimal layout, or topology of a structure. Topology optimisation is the most general form of structural optimisation and is concerned with finding the boundary of a given problem domain which is optimal in that it minimises an objective functional while satisfying given constraints.

In this paper a topology optimisation is implemented by performing an adjoint-based sensitivity analysis to compute the gradient of the objective functional as described in [1]. The level set function (LSF) which defines the internal boundaries of the problem can then be advected along the steepest descent gradient towards an optima. The LSM developed by Osher and Sethian [2] is a simple and efficient method for computing the evolution of a moving interface. Since it was first used for structural optimisation in [4], the LSM has been used for boundary tracking in many shape optimisation implementations and has proven efficacy.

In this paper, the spatial domain of both the physical problem and the level set transport problem are discretised using the DGSIP method [3]. The DG method differs from the continuous Galerkin (CG) method in that it works over a trial space of functions which are only piecewise continuous. One of the main advantages of DG finite elements is that they are trivially parallelisable. As such this work is a first step towards exploiting parallel computing techniques to create an efficient shape optimisation method for large-scale problems.

Following this introduction, Section 2 will present and explain an algorithm for finding the solution to the minimum compliance problem for a cantilever beam. This will be followed by some numerical results for the given problem and conclusions in Section 3.

2. Topology optimisation algorithm and problem formulation

For the general case the proposed optimisation algorithm is as follows:

1. Define the problem domain.
2. Initialise the LSF as a signed distance function (SDF).
3. Until the value of the objective functional converges:
    (a) compute the current state and the adjoint state through the solution of the physical and adjoint problems;

(b) compute the advection velocity; and
(c) evolve the interface through the solution of the transport Hamilton-Jacobi equation.
  i. Reinitialise the LSF to a SDF through a geometric redistancing method.

The algorithm will now be explained through the use of an example optimisation problem. The objective of the optimisation process will be to determine the boundary, which for a given domain, load distribution and boundary conditions, will minimise the compliance of the system whilst satisfying a given weight constraint. Mathematically this can be stated as:

$$\inf J(\Omega) \quad \text{where:} \quad J(\Omega) = \int_{\partial \Omega_N} g \cdot u \, ds + \ell \int_{\Omega} dx \tag{1}$$

Where J is the objective functional, Ω is the domain, $\partial \Omega_N$ is the Neumann part of the boundary, g are the surface tractions, u is the displacement field, and ℓ is a penalty term enforcing the weight constraint.

2.1. Step 1: Defining the problem domain

The problem to be discussed in this paper aims to find the optimal topology of a 2D cantilever beam problem, the dimensions and the boundary/loading conditions are shown in Figure 1(a). The domain is discretised into a mesh of 100 by 50 first order, piecewise continuous, square elements. The domain has two sets of elastic properties which represent the material section and the void section. The material part of the domain has Young's modulus, $E = 1$Pa and Poisson's ratio, $v = 0.3$, and the void is modelled as a much more compliant material section which has a Young's modulus $E = 10^{-3}$Pa and Poisson's Ratio, $v = 0.3$. The value of Young's modulus varies smoothly across the interface between the two materials by applying a smoothed sign function to the LSF and using this as the input to a linear function which varies between the two values of Young's Modulus as the sign function varies between -1 and 1.
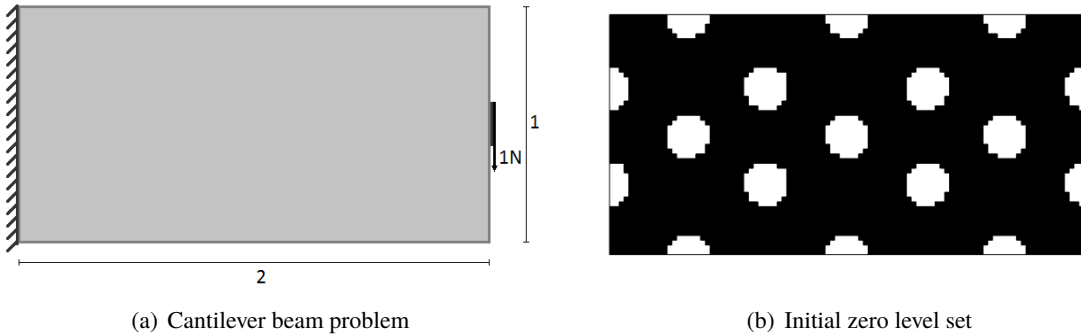

(a) Cantilever beam problem

(b) Initial zero level set

Figure 1: Initialisation of the problem

2.2. Step 2: Initialisation of the LSF as an SDF

The interface between the material portion and the void portion of the domain is defined as the zero level set of the LSF, ϕ. Where the level set function is greater than zero the domain is filled with material and where the level set function is less than zero the domain is filled with the ersatz material which represents the void as further described in Section 2.1.

There are two main reasons for initialising the LSF as an SDF. First of all, numerical inaccuracies will occur during the transport process if there are large variations in the gradient of the LSF, especially in the region close to the zero level set. Secondly, initialising the LSF as an SDF and maintaining this property through frequent reinitialisation greatly simplifies the level set transport equation as discussed in Section 2.5.

When choosing the initial LSF, one must be sure to ensure that there are sufficient holes in the domain. This is because for a two dimensional problem being evolved with a time stepping scheme which satisfies a strict CFL condition, no holes can be created during the optimisation process [1]. The initial configuration can be seen in Figure 1(b), where black represents the material and white represents the voids.

2.3. Step 3(a): Computation of the state and the adjoint state through DG solution of the physical model

In order to compute the gradient of the objective functional, an adjoint-based sensitivity analysis is performed. The minimum compliance problem is self-adjoint and as such is simpler than the general case in which an adjoint state would also have to be computed at each iteration. The author points the reader towards [1] for the details of deriving the topological derivative and simply states the result that for the objective functional stated in (1) the topological derivative yields a vector field, V:

$$V = bn \quad \text{with} \quad b = \sigma(u) \cdot \varepsilon(u) - \ell \tag{2}$$

In order to compute this advection velocity field, it is necessary to first compute the displacement field for the current state of the cantilever beam problem. For a linear elastic stress analysis problem, the DG approximation for the displacement field, u, is given by:

$$\int_{\Omega} f v d\Omega = \int_{\Omega} vB \cdot DBu \, d\Omega - \int_{\Gamma} [\![v]\!] \cdot D\{\!\{Bu\}\!\} \, d\Gamma - \int_{\Gamma} \{\!\{Bv\}\!\} \cdot D[\![u]\!] \, d\Gamma + \int_{\Gamma} \mu[\![v]\!] \cdot [\![u]\!] \, d\Gamma \tag{3}$$

Where Γ denotes the element face, f are the body forces, v is the test function, B is the strain-displacement matrix, D is the elastic properties matrix, μ is a penalty parameter, $[\![q]\!]$ denotes the jump of q and $\{\!\{q\}\!\}$ denotes the average of q. For this problem, the boundary conditions are imposed strongly. Once again the author directs the reader to [3], for the details on the discretisation of elliptic problems using the DGSIP method.

2.4. Step 3(b): Computation of the advection velocity

The advection occurs only in the normal direction to the interface and so the advection velocity at each node is given by, b, defined in (2). The stress, σ, and strain, ε, can be recovered from the displacement solution of the linear elasticity problem described in Section 2.3. The ℓ term is a penalty term which describes the volume ratio, i.e. the ratio of material to void, at convergence. It is prescribed during the initial iteration, for the numerical example in this paper, $\ell = 150$.

2.5. Step 3(c): Evolution of the interface

The evolution of the LSF occurs through the solution of a Hamilton-Jacobi equation. However, as mentioned in Section 2.2, the maintenance of the LSF as an SDF ensures that the Euclidean norm of the spatial derivatives is always equal to unity and as such the level set transport equation can be greatly simplified as described in [5], which can be stated as follows:

$$\frac{\partial \phi}{\partial t} = b \tag{4}$$

Equation (4) is solved explicitly in time using a forward Euler method. The time step is constrained by the CFL condition and is inversely dependent on the maximum absolute value of the advection velocity field at each iteration and the size of the elements in the mesh. (4) is solved in space using DG finite elements once again. For this particular equation, both CG and DG formulations are the same, however, the DG formulation of (4) can be solved independently for each element, allowing for a much more efficient solution.

2.6. Step 3(c)i: Reinitialisation of the level set function

Maintaining the signed distance property of the LSF can be achieved through multiple methods. One method is known as geometric redistancing (or the brute force method). For this Yamasaki *et al.* [5] presented a method which involves discretising the zero level set and then computing the minimum distance from each node to the discretised interface and multiplying this by the sign of the level set at that node. When using a discontinuous LSF the multiplication by the sign can cause discontinuities at the interface which results in the LSF not being reinitialised as an SDF. To remedy this a smoothing is applied to the LSF prior to the reinitialisation whereby the value of the LSF at each node is taken to be the average of the values at each of the degrees of freedom at that node.

3. Numerical results and conclusions

The problem presented in the previous section was ran for 50 iterations. Figure 2(a) shows a smooth convergence of the objective functional until it reaches a minimum. The final shape after 50 iterations is shown in Figure 2(b).

(a) Convergence of the objective functional

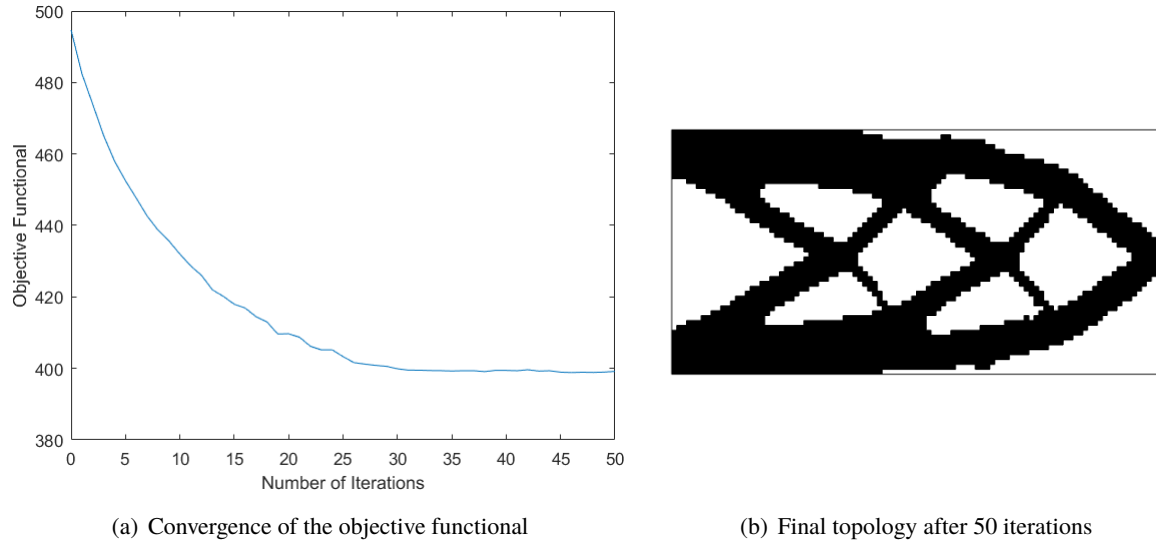(b) Final topology after 50 iterations

Figure 2: Numerical results

To conclude, a topology optimisation algorithm using level set methods and the discontinuous Galerkin method has been presented and explained. The algorithm was implemented to find the optimal topology of a 2D linear elastic cantilever beam. The results show that the algorithm has good convergence and stability properties, and the optimal design shown is similar to that published in the literature.

Acknowledgements

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