Proceedings of the 24th UK Conference of the
Association for Computational Mechanics in Engineering
31 March - 01 April 2016, Cardiff University, Cardiff

A Model Reduction Method for Boundary Element Method

***Shengze Li[1], Jon Trevelyan[2] and Weihua Zhang[1]**

[1]College of Aerospace Science and Engineering, National University of Defense Technology, 109 Deya Road, Changsha 410073, China
[2]School of Engineering and Computing Sciences, Durham University, South Road, Durham DH1 3LE

*lishengze12@gmail.com

ABSTRACT

The aim of this work is to provide a promising way to improve the computational efficiency for BEM. This study introduces an 'a priori' model reduction method in BEM analysis aiming to enhance efficiency by approximating the problem solution using the most appropriate set of basis functions, which depend on Karhunen-Loève decomposition. The calculation process will proceed making use of a precomputed basis function space if the norm of the residual is small enough; if not, it need to enrich the approximation basis and compute again some of the previous steps. Finally, an example is proposed which demonstrates fast resolution of BEM problem and illustrates the potential of this numerical technique. This work is preliminary work in a larger programme leading to optimisation using isogeometric BEM scheme accelerated with the Proper Orthogonal Decomposition.

Key Words: *BEM; model reduction; Karhunen-Loève decomposition*

1. Introduction

The Boundary Element Method (BEM) [1] is a domain discretization technique for solving partial differential equations; one major advantage of boundary integral equation approaches is to decrease the dimension by one, *i.e.*, only line integrals are required in 2D and surface integrals in 3D problems, from which a smaller system of equations will be generated. Hence, this alleviates the burden of mesh generation as a surface mesh generation is much easier, more rapid and (importantly) more robust than domain mesh. In spite of teh advances in FE mesh generation, these advantages of BEM remain considerable when the objective is to analyse many, similar geometries, such as in optimisation schemes, involving multiple remeshing operations. The drawbacks of BEM include the fully populated matrix structure; it also dos not lend itself as well as FEM to materially non-linear problems. In order to accelerate the solution (specifically, the re-solution) of the dense linear system, model reduction techniques present a promising strategy.

Model reduction techniques have been derived from some problems in random data processing, and further used in image processing. They have been successfully applied in some finite element frameworks [2, 3], the nonlinear mechanical problem [4, 5] and 2D BEM in fluid mechanics [6]. In the current work we assess their suitability for the BEM analysis process in 3D elasticity, updating solutions following geometric changes that might occur during an optimisation process. In the future this will be extended into an isogeometric BEM context.

1.1. Boundary Element Discretization

For a linear elastic problem, the structure occupies a continuous domain Ω, having boundary Γ, with the boundary conditions,

$$\begin{aligned} u &= \overline{u} \quad \text{on } \Gamma_u \\ t &= \overline{t} \quad \text{on } \Gamma_t \end{aligned} \tag{1}$$

where the domain boundary $\Gamma = \Gamma_u + \Gamma_t$.

The Boundary Integral Equation (BIE) can be written as follows,

$$u_i(s) + \int_\Gamma T_{ij}(s,x)u_j(x)d\Gamma(x) = \int_\Gamma U_{ij}(s,x)t_j(x)d\Gamma(x) \tag{2}$$

where s is the 'source' point and x the 'field' point. By using standard methods of discretisation of (2) and taking s as all nodes in turn, we arrive at the matrix form

$$\underline{\underline{H}}\,\underline{u} = \underline{\underline{G}}\,\underline{t} \tag{3}$$

where $\underline{u}$ contains the nodal displacements and $\underline{t}$ the nodal tractions. Both $\underline{u}$ and $\underline{t}$ include a combination of unknown values and prescribed boundary conditions. Rearrangement of (3) leads to the final form

$$\underline{\underline{A}}\,\underline{x} = \underline{F} \tag{4}$$

where the size of $\underline{\underline{A}}$ is $N \times N$.

1.2. The Karhunen-Loève Decomposition (KLD)

Also known as Proper Orthogonal Decomposition (POD), the KLD is a powerful and elegant method for data analysis aimed at obtaining low-dimensional descriptions of a high-dimensional process.

For an arbitrary evolution process, a certain field could be defined by a discrete form as $u^p(\underline{x}_i)$, $\forall p \in [1,\ldots,P], \forall i \in [1,\ldots,N]$, describing the nodal displacement $(\underline{x}_i)$ at the calculation step p. The main idea of the KLD is to obtain a low dimensional space containing the most typical or characteristic behaviour among the displacement fields. This is equivalent to obtaining a function $\phi(\underline{x})$ maximizing α defined by

$$\alpha = \frac{\sum_{p=1}^{P}\left[\sum_{i=1}^{N}\phi(\underline{x}_i)u^p(\underline{x}_i)\right]^2}{\sum_{i=1}^{N}(\phi(\underline{x}_i))^2} \tag{5}$$

Introducing a vector notation, Eq.(5) takes the following matrix form

$$\underline{\underline{k}}\,\underline{\phi} = \alpha\underline{\phi} \tag{6}$$

The two point correlation matrix $\underline{\underline{k}}$ is given by

$$\underline{\underline{k}} = \sum_{p=1}^{P}\underline{u}^p(\underline{u}^p)^{\mathrm{T}} \tag{7}$$

which is symmetric and positive definite.

Here, the functions defining the characteristic structure of $u^p(\underline{x})$ are the eigenfunctions $\phi_k(\underline{x}) \equiv \underline{\phi}_k$ associated with the highest eigenvalues.

2. Reduced Model Construction

If some direct simulations have been carried out previously, the nodal displacement can be defined as $u(\underline{x}_i, s_p) \equiv u_i^p$, $\forall i \in [1,\ldots,N], \forall p \in [1,\ldots,P]$. The eigenvalues are assumed ordered, if $\alpha_k > 10^{-10}\alpha_1$, $\forall k \in [1,\ldots,n]$, ($\alpha_1$ is the highest eigenvalue). Then, those n eigenvectors related to the eigenvalues above could be used for generating an approximating basis for further solutions. The matrix $\underline{\underline{B}}$ is defined as

$$\underline{\underline{B}} = \begin{bmatrix} \phi_1^1 & \phi_1^2 & \cdots & \phi_1^n \\ \phi_2^1 & \phi_2^2 & \cdots & \phi_2^n \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^1 & \phi_N^2 & \cdots & \phi_N^n \end{bmatrix} \tag{8}$$

where N is the DOF of the system and we take eigenvectors 1,...,n. The nodal displacement vector in Eq.(4) could be written as

$$\underline{x} = \sum_{i=1}^{n} \zeta_i \underline{\phi}_i = \underline{\underline{B}}\underline{\zeta} \tag{9}$$

which could be substituted to Eq.(4) to obtain

$$\underline{\underline{A}}\underline{\underline{B}}\underline{\zeta} = F \tag{10}$$

and we finally premultiply both sides by $\underline{\underline{B}}^{\mathrm{T}}$,

$$\underline{\underline{B}}^{\mathrm{T}}\underline{\underline{A}}\underline{\underline{B}}\underline{\zeta} = \underline{\underline{B}}^{\mathrm{T}}F \tag{11}$$

This procedure provides a final $n \times n$ matrix in a low dimension. While the generation of matrix $\underline{\underline{A}}$ is still time consuming, it can be accelerated by er-use of matrix coefficients that are unchanged from a previous design iteration. This process is likely to be made more powerful and general when the authors implement their future plans of bringing POD to an isogeometric BEM context.

3. An 'a priori' Model Reduction Strategy

As the geometry evolves during the design/optimisation process, the basis matrix B should be updated. The basis matrix $\underline{\underline{B}}$ is generated by the first P steps, and another S steps will be analysed with this basis. After each S steps, the residual of the system should be evaluated as

$$\underline{R} = \underline{\underline{A}}\underline{x} - \underline{F} = \underline{\underline{A}}\underline{\underline{B}}\underline{\zeta} - \underline{F} \tag{12}$$

If the norm of residual is sufficiently small, $\| \underline{R} \| < \epsilon$, with ϵ a threshold value, the next S steps will be continued; otherwise, the approximation basis should be enriched and the last S steps recomputed to ensure accuracy is maintained. The enrichment is built using the Krylov subspace, the new basis matrix being defined as

$$\underline{\underline{B}}^{*} = \{\underline{\underline{B}}\underline{\underline{V}}, \underline{R}, \underline{\underline{A}}\underline{R}, \underline{\underline{A}}^{2}\underline{R}\} \tag{13}$$

where $\underline{\underline{V}}$ is the combination of the most representative eigenvectors which is from the previous reduced result $\underline{\zeta}$. A new solution vector could be written as

$$\underline{\zeta}^{*} = \left[\underline{\underline{B}}^{*\mathrm{T}}\underline{\underline{B}}^{*}\right]^{-1} \underline{\underline{B}}^{*\mathrm{T}}\underline{\underline{B}}\underline{\zeta} \tag{14}$$

4. Numerical Example

In this preliminary study, the problem is defined as a simple cube (Fig. 1) which is under a uniaxial compressive pressure of 1MPa in the z-direction. The left, back and bottom surfaces have normal displacement constraints. The material properties of steel are used. As a design evolution process, the height h will be continuously increased in 0.02m increments. This problem is solved first $P = 25$ steps for generating the approximation matrix using the conventional BEM approach. The approximation is performed using 54 elements, giving 168 degrees of freedom. The eigenvalues and eigenvectors are derived from those pre-calculated displacements, and only 4 eigenvalues satisfy the selection criterion ($\alpha_k > 10^{-10}\alpha_1$). This means a 168×168 matrix will be reduced to a 4×4 matrix for solving the remaining design iterations. The next calculation step will be divided to $S = 35$ parts, for each part, including $S_{sub} = 5$ sub-calculation steps. In this case, the solution is performed starting from the reduced basis which is obtained previously and after each 5 steps, the quality of the solution is checked and follows the previous criterion for judging whether the basis should be enriched.

Fig. 2 compares the displacement result of the top surface in the z-direction; as the figure shows, the reduction model provides an accurate result with a lower dimensional computation, and the error is within 0.14%.

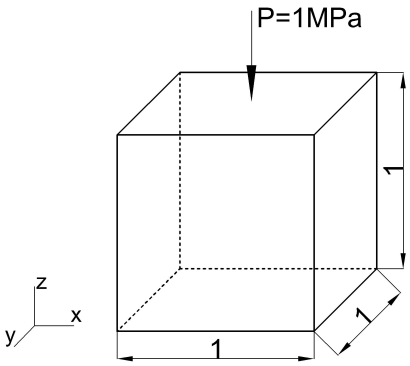

Figure 1: The cube under uniaxial compression

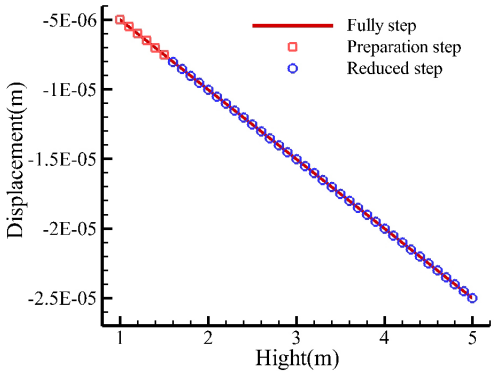

Figure 2: Displacement solutions

5. Conclusions

In this article, the Karhunen-Loève decomposition model reduction approach is combined with the Boundary Element Method for 3D elasticity. The size of linear system is dramatically decreased while mainitaining appropriate accuracy. The numerical example proves the potential of this method. Further work will demonstrate the extension of this method to isogeometric BEM, and linking with the optimisation process for more complicated structural components.

References

[1] T. A. Cruse. Numerical solutions in three dimensional elastostatics. International Journal of Solids and Structures, 5, 1259–1274, 1969.

[2] D. Ryckelynck. A priori hyperreduction method: an adaptive approach. Journal of Computational Physics, 202, 346-366, 2005.

[3] D. Ryckelynck, F. Chinesta, E. Cueto, A. Ammar. On the a priori model reduction: overview and recent developments. Archives of Computational Methods in Engineering, 13, 91-128, 2006.

[4] K. Carlberg, C. Bou-Mosleh, C. Farhat. Efficient non-linear model reduction via a least-squares Petrov-Galerkin projection and compressive tensor approximations. International Journal for Numerical Methods in Engineering, 86, 155-181, 2011.

[5] P. Kerfriden, P. Gosselet, S. Adhikari, S. Bordas. Bridging proper orthogonal decomposition methods and augmented Newton-Krylov algorithms: An adaptive model order reduction for highly nonlinear mechanical problems. Computer Methods in Applied Mechanics and Engineering, 200, 850-866, 2011.

[6] D. Ryckelynck, L. Hermanns, F. Chinesta, E. Alarcon. An efficient 'a priori' model reduction for boundary element models. Engineering Analysis with Boundary Elements, 29, 796-801, 2005.