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An Isogeometric Boundary Element Method with Subdivision Surfaces for Helmholtz Analysis

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ABSTRACT

The present work develops a new isogeometric Boundary Element Method with subdivision surfaces for solving Helmholtz problems. The work gives a brief overview of subdivision surfaces and their use in an analysis context highlighting the pertinent points for boundary element analysis. We find that by adopting the high order (quartic) basis functions of subdivision surfaces to perform Helmholtz analysis a higher accuracy per degree of freedom is obtained over equivalent Lagrangian discretisations. We demonstrate this through a Helmholtz problem with a closed-form solution in which a plane wave is impinged on a 'hard' sphere.
Key Words: *Isogeometric analysis; boundary element method; Helmholtz; subdivision surfaces*

1. Introduction

Isogeometric analysis (IGA) has rapidly expanded in recent years into a major research effort to link Computer Aided Design (CAD) and numerical methods driven by the need for more effcient industrial design tools. The central idea of IGA is that the same discretisation model is used for both design and analysis which eliminates costly model conversion processes encountered in traditional engineering design work flows. IGA was originally conceived by Hughes et al. [1] and has predominately focussed on the use of the Finite Element Method, but work has also applied the approach to the Boundary Element Method (BEM) where distinct advantages are found, stemming from the need for only a surface mesh. In the majority of IGA implementations the most commonly used CAD discretisation is the Non-Uniform Rational B-Spline (NURBS) due to its ubiquitous nature in CAD software. However, NURBS technology has limitations due to its tensor-product nature and a number of researchers have developed alternative CAD discretisations which overcome this limitation. One such example is T-spline technology which has been employed in an IGA setting by Bazilevs et al. [2] in 2010. Subdivision surfaces provide another alternative for overcoming the limitations of NURBS, initially introduced by Cirak [3] in 2000. The present work have developed an isogeometric Boundary Element Method with subdivision surfaces for solving acoustic problems.

2. Boundary Element Method for Helmholtz Problems

For time-harmonic acoustic problems, the governing equation is the Helmholtz equation which is expressed as

$$\nabla^2\phi(\mathbf{x}) + k^2\phi(\mathbf{x}) = 0 \tag{1}$$

where $\phi(\mathbf{x})$ is the acoustic pressure and k is the wavenumber defined as $k = \frac{\omega}{c}$ where ω is the angular frequency and c is the thermodynamic speed of sound. Several numerical methods can be used to solve Eq. (1) but many suffer from problems including dispersion error and difficulties in handling infinite domains (e.g. the finite element method). In contrast, the Boundary Element Method does not suffer from dispersion error and naturally handles infinite domains. A core feature of the BEM is the use of fundamental solutions to arrive at a numerical solution, and in the case of 3D acoustic problems the fundamental solution is given by:

$$G(\mathbf{x}, \mathbf{y}) = \frac{e^{ikr}}{4\pi r} \tag{2}$$

where $\mathbf{x}$ is the source point, $\mathbf{y}$ is the field point and $r = |\mathbf{x} - \mathbf{y}|$. The corresponding normal derivative of this kernel can be expressed as:

$$\frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n} = \frac{e^{ikr}}{4\pi r^2}(ikr - 1)\frac{\partial r}{\partial n}. \tag{3}$$

The use of Eq. (2) and (3) allows a boundary integral equation to be formulated that relates acoustic pressure and its normal derivative as:

$$c(\mathbf{x})\phi(\mathbf{x}) + \int_s \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n}\phi(\mathbf{y})\,\mathrm{d}S(\mathbf{y}) = \int_s G(\mathbf{x}, \mathbf{y})\frac{\partial \phi(\mathbf{y})}{\partial n}\,\mathrm{d}S(\mathbf{y}) + \phi_{inc}(\mathbf{x}) \tag{4}$$

where S represents the boundary surface, ϕ_{inc} denotes the acoustic pressure of a prescribed incident wave (only applicable in the case of scattering problems) and $c(\mathbf{x})$ is a coefficient that depends on the geometry of the surface at the source point. In the case of smooth boundaries, its value is given by $c(\mathbf{x}) = \frac{1}{2}$.

For computer implementation purposes, the acoustic pressure and normal derivative are discretised through an appropriate set of basis functions as:

$$\phi(\mathbf{x}) = \sum_{A=1}^{n} \phi_A N_A(\mathbf{x}) \qquad \frac{\partial \phi(\mathbf{x})}{\partial n} = \sum_{A=1}^{n} q_A N_A(\mathbf{x}) \tag{5}$$

where in the present study $\{N_A(\mathbf{x})\}_{A=1}^{n}$ is a global set of subdivision basis functions, $\{\phi_A(\mathbf{x})\}_{A=1}^{n}$ is a set of nodal acoustic pressure coefficients and $\{q_A(\mathbf{x})\}_{A=1}^{n}$ is a set of nodal acoustic velocity coefficients.

3. Subdivision Surfaces

There exist a variety of subdivision schemes, but all are based on the idea of generating a smooth surface from a coarse polygon mesh. Subdivision refinement schemes construct a smooth surface through a limiting procedure of repeated refinement starting from an initial control mesh. Subdivision refinement schemes can be classed as either an interpolating or approximation scheme where Figure 3 illustrates the commonly used Catmull-Clark approximation scheme.

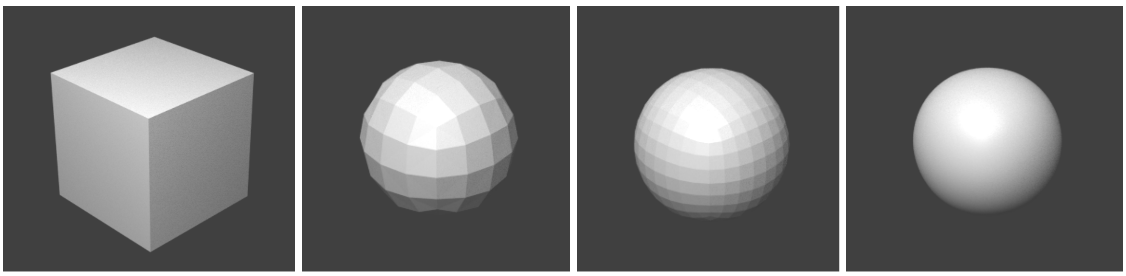

Figure 1: Successive levels of Catmull-Clark subdivision refinement applied to an initial cube control mesh.

Subdivision basis functions can be used as a basis for analysis [5] and have successfully been used in variety of applications including thin-shell finite element analysis [3]. We extend the work of [3] by adopting a Loop subdivision scheme and applying subdivision basis functions for acoustic boundary element analysis.

The Loop subdivision scheme is based on a triangular tessellation with a typical element and its one-ring neighbours shown in Figure 2 corresponding to a regular patch where each vertex has 6 connecting edges.

The shaded element in this patch contains 12 non-zero quartic basis functions as detailed in [5]. In the present study these basis functions are used to dicretise the acoustic pressure and acoustic velocity. In addition, we apply a collocation procedure whereby a set of global collocation points is generated through interpolation of the limit surface at vertex locations allowing a system of equations to be constructed.

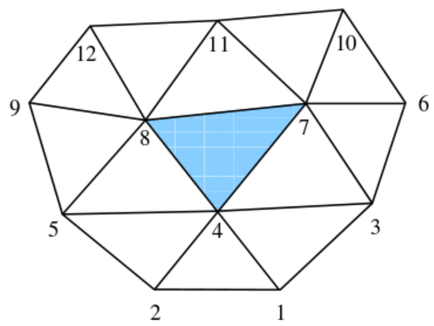

Figure 2: An element defined over a regular patch with 12 control points

4. Results

To validate the current subdivision BEM the approach was applied to the problem of a 'hard' sphere impinged by a plane-wave, as illustrated in Figure 3. In this problem a dimensionless wavenumber of $k/a = 8$ was chosen with the acoustic pressure sampled at a set of points surrounding the sphere.

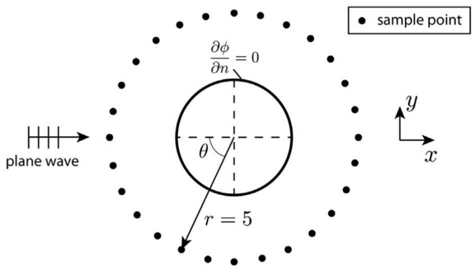

Figure 3: Problem of a plane wave impinged on a 'hard' sphere [4]

Two sets of boundary element analysis were performed for this problem: the first applied a conventional Lagrangian discretisation using quartic basis functions defined over triangular elements and the second applied quartic subdivision discretisations. In the study, a Lagrangian mesh with 1152 degrees of freedom (dof) and two subdivision meshes with 438 and 1746 dof were used.

The results for the three discretisations at each of the sampling points are shown in Figure 4 along with the analytical solution. A a closeup view of this plot illustrates that both subdivision discretisations deliver higher accuracies compared to the Lagrangian discretiation even with fewer degrees of freedom.

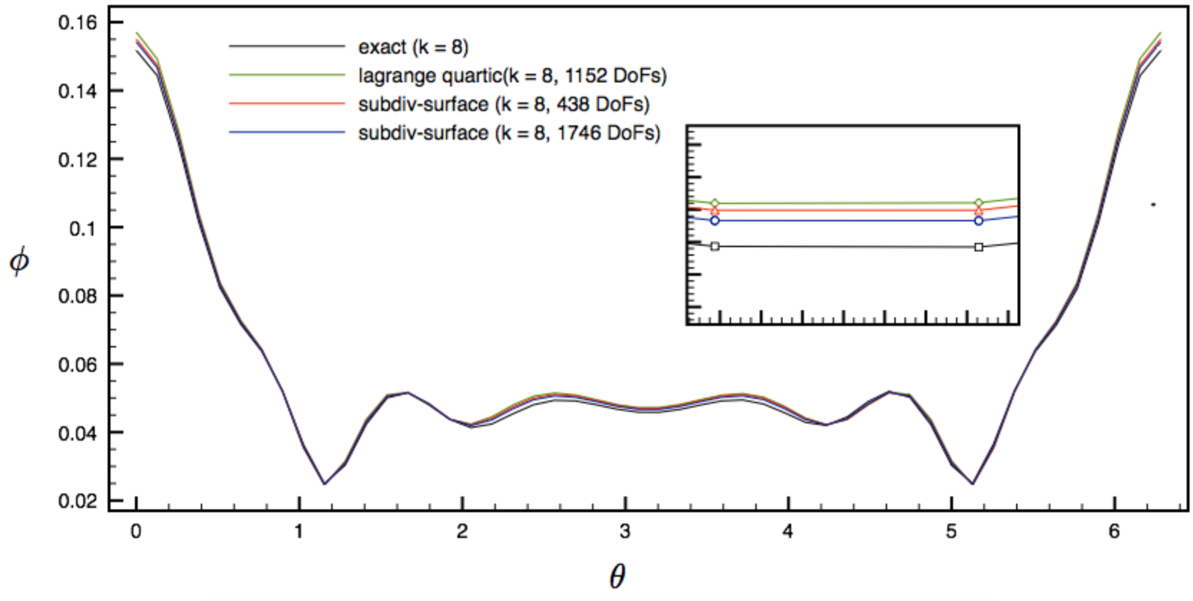

Figure 4: A comparison between Lagrangian discretisations and subdivision surfaces (k = 8)

5. Conclusions and Future Research

The present work develops a new isogeometric Boundary Element Method with subdivision surfaces for solving Helmholtz problems where it is found that higher accuracies are obtained over a Lagrangian discretisation of the same order. There are several future research directions for the present work:

- Coupling the Boundary Element Method and the Finite Element Method for structural-acoustic analysis
- Simulating electromagnetic scattering problems by developing suitable div- and curl-conforming discretisations

References

[1] T.J.R. Hughes, J.A. Cottrell, and Y. Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Computer Methods in Applied Mechanics and Engineering*, 194(39):4135-4195, 2005.

[2] Y. Bazilevs, V.M. Calo, J.A. Cottrell, J.A. Evans, T.J.R. Hughes, S. Lipton, M.A. Scott, and T.W. Sederberg. Isogeometric analysis using T-splines. *Computer Methods in Applied Mechanics and Engineering*, 199(5-8):229-263, 2010.

[3] F. Cirak, M. Ortiz, and P. Schroder. Subdivision surfaces: a new paradigm for thin-shell finite-element analysis. *International Journal for Numerical Methods in Engineering*, 47(12):2039-2072, 2000.

[4] R.N. Simpson, M.A. Scott, M. Taus, D.C. Thomas, and H. Lian. Acoustic isogeometric boundary element analysis. *Computer Methods in Applied Mechanics and Engineering*, 269:265-290, 2014.

[5] J. Stam. Exact evaluation of Catmull-Clark subdivision surfaces at arbitrary parameter values. *Proceedings of the 25th annual conference on Computer graphics and interactive techniques*, 395–404, ACM, 1998.